Wadi Alshatti University Journal of Pure and Applied Sciences

مجلة جامعة وادي الشاطئ للعلوم البحتة والتطبيقية

Volume 4, No. 1, January-June 2026 Online ISSN: 3006-0877 المجلد 4، الاصدار 1، يناير- يونيو 2026

RESEARCH ARTICLE NUMERICAL ANALYSIS

Bernstein Polynomials for Solving Some Classes of Abel's Integral Equations with Weakly-Singular Kernels

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ARTICLE HISTORY

Received 01 March 2026
Revised 28 March 2026
Accepted 04 April 2026
Online 07 April 2026

KEYWORDS

Abel's integral equations;
Generalized Abel's integral equations;
Main generalized Abel's integral equations;
Weakly-singular Integral Equation;
Bernstein polynomials.

ABSTRACT

Most physical phenomena are described by differential equations, such equations are frequently hard to solve in a direct way, they are commonly converted into integral equations, which can then be tackled with simpler and more efficient methods. This paper aims to numerically obtain numerical solutions of the linear main generalized Abel's integral equations (weakly-singular kernel) of both first and second kinds using the Bernstein polynomials where the unknown function is approximated in terms of such polynomials. Then relying on some properties of the Bernstein polynomials, the considered integral equation is converted into a linear system of algebraic equations, that can be easily solved to obtain the coefficients of expansion. To evaluate its performance, some examples were presented comparing the numerical solutions with the exact ones. These comparisons showed that, even for low-degrees Bernstein polynomials, the approximate solution agrees very well with the exact one. The obtained results confirm the method's effectiveness and dependability for solving integral equations with weakly-singular kernels. Additionally, the numerical findings ensure the analytical theorems concerning existence, uniqueness, and convergence of the numerical solution.

متعددات حدود بيرنشتاين لحل بعض أصناف معادلات آبل التكاملية الخطية ذات نواة ضعيفة الاعتلال

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الكلمات المفتاحية

معادلات آبل التكاملية الخطية
معادلات آبل التكاملية المعممة
معادلات آبل التكاملية المعممة الرئيسية
المعادلات التكاملية ذات الاعتلال الضعيف
متعددات حدود بيرنشتاين

الملخص

توصف معظم الظواهر الفيزيائية بالمعادلات التفاضلية، وغالبًا ما يكون من الصعب حل هذه المعادلات مباشرة، لذلك تُحول عادةً إلى معادلات تكاملية مكافئة، والتي يمكن معالجتها لاحقًا باستخدام أساليب أبسط وأكثر كفاءة. تهدف هذه الورقة الى الحصول على الحلول التقريبية لمعادلات أبل التكاملية الخطية المعممة ومعادلات آبل التكاملية المعممة الرئيسية. في هذا الورقة تم استخدام كثيرات الحدود بيرنشتاين والتي تعتمد على تقريب الدالة المجهولة كتركيبة خطية من كثيرات حدود بيرنشتاين. وباستخدام خصائص هذه الكثيرات يتم اختزال المعادلة التكاملية إلى نظام من المعادلات الجبرية الخطية، والذي يمكن حلها بسهولة لإيجاد معاملات كثيرات حدود بيرنشتاين. ولتقييم أدائها، عُرضت بعض الأمثلة التي قورنت فيها الحلول العددية المتحصل عليها مع الحلول التحليلية المضبوطة. أظهرت هذه المقارنة أنه حتى عند درجات صغيرة من كثيرات حدود برنشتاين، تتقارب الحلول التقريبية إلى الحلول المضبوطة. تؤكد النتائج فعالية وموثوقية هذه الطريقة في حل المعادلات التكاملية. بالإضافة إلى ذلك، تُدعم النتائج العددية النظريات التحليلية المتعلقة بوجود ووحدانيةالحلّ التقريبي وتقاربه.

Introduction

Integral equations are widely recognized as one of the most significant mathematical tools utilized in both pure and applied mathematics. These equations establish a crucial connection between a function and its integral, allowing for sophisticated problem-solving and analysis across various fields [1,2]. With the significant advances in various sciences and the growing complexity arising from their interactions and rapid development, researchers have begun to study physical, chemical, and engineering phenomena more deeply. Integral equations of diverse kinds have been crucial in understanding these phenomena and solving them in an effective manner.

In this paper, we consider the Linear Generalized Abel's Integral Equations (LGAIEs) of both the first and second kinds respectively as,

$$f(x) = \int_0^x \frac{1}{|x-t|^\rho}\varphi(t)dt. \quad (1)$$

And,

$$\varphi(x) = f(x) + \int_0^x \frac{1}{|x-t|^\rho}\varphi(t)dt \quad (2)$$

where $f(x)$ is a given function, $\varphi(x)$ is the unknown function, and ρ is a real constant such that $0 < \rho < 1$. It

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https://doi.org/10.63318/waujpasv4i1_34

should be noted that, the well-known Abel integral equation is a special case of the generalized equation for $\rho = \frac{1}{2}$ [1].

Moreover, the Linear Main Generalized Abel's Integral Equation (LMGAIEs) is given by the following form,

$$f(x) = \int_0^x \frac{1}{|g(x) - g(t)|^\rho} \varphi(t)dt, \quad 0 < \rho < 1,$$

where $g(t)$ is strictly monotonically increasing and differentiable function in some interval $[a, b]$, and $g'(t) \neq 0$ for every t in the interval.

Abel's integral equation is acknowledged as the earliest type of integral equation. A variety of applications depend on this equation, namely X-ray imaging, scattering theory, earth sciences, radar measurements, and optical fibre assessment [3,4]. Recently, there is a considerable amount of research related to the polynomial-based method for solving the LMGAIEs. For instance; the Laguerre polynomials were implemented for solving Volterra integral equation of singular kernel [5]. The Taylor-collocation method was used to solve the LGAIEs of both the first and second kinds [6]. Also, the Bernoulli polynomials was implemented for solving integral equations with non-regular kernel [7].

The structure of this paper is as follows: The related literature was reviewed in the introduction section. Then the definition of the Bernstein polynomials with some of its important properties and some of their recursive formulas were introduced in section one. The second section; considers the formulation of the numerical solution of the LGAIEs by the Bernstein polynomials. In the third section we recall some important theorems related the existence, uniqueness and the convergence of the obtained numerical solution of LGAIEs. To verify the considered method, some numerical examples are shown in section four followed by a discussion and conclusion in section five.

Bernstein Polynomials

The Bernstein polynomials of n^{th} degree on interval [0,1] is defined by the following form:

$$B_{i,n}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \tag{3}$$

Where,

$$\binom{n}{i} = \frac{n!}{i!\,(n-i)!}, i = 0,1,2,\dots,n.$$

The n^{th} degree Bernstein polynomials on the interval $[a, b]$ are defined as [8]:

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, i = 0,1,2,\dots,n.$$

For n = 5, These polynomial basis functions over interval [0, 1] are given as follows:

$B_{0,5}(x) = (1-x)^5$,

$B_{1,5}(x) = 5x(1-x)^4$,

$B_{2,5}(x) = 10x^2(1-x)^3$,

$B_{3,5}(x) = 10x^3(1-x)^2$,

$B_{4,5}(x) = 5x^4(1-x)$,

$B_{5,5}(x) = x^5$.

The Bernstein polynomials have some properties are described below [9]

1- The Bernstein Polynomials are all non-negative.
2- $\sum_{i=0}^{n} B_{i,n}(x) = 1$
3- For $i, j = 0,1,2,\dots,n$, we have
$$B_{i,n}(x)B_{j,m}(x) = \frac{\binom{n}{i}\binom{m}{j}}{\binom{n+m}{i+j}} B_{i+j,n+m}(x).$$
4- $xB_{i,n}(x) = \frac{i+1}{n+1} B_{i+1,n+1}(x)$.
5- $\int_a^b B_{i,n}(x)\,dx = \frac{(b-a)}{n+1}$.
6- $\int_a^b B_{i,n}(x)\,B_{k,n}(x) = \frac{(b-a)}{(2n+1)} \frac{\binom{n}{i}\binom{n}{k}}{\binom{2n}{i+k}}\ i, k = 0,1,\dots,n$ (4)

The Bernstein polynomials should satisfy the following recursive formulas:

$B_{i,n}(x) = (1-x)B_{i,n-1}(x) + xB_{i-1,n-1}(x)$,

$B_{i,n-1}(x) = \left(\frac{n-i}{n}\right) B_{i,n}(x) + \left(\frac{n+i}{n}\right) B_{i+1,n}(x)$,

$B'_{i,n}(x) = \frac{d}{dx}\left(B_{i,n}(x)\right) = n\left[B_{i-1,n-1}(x) - B_{i,n-1}(x)\right]$,

$B_{i,n}(1-x) = B_{n-i,n}(x)$.

Numerical Solution of the LMGAIEs by using Bernstein Polynomials

We consider the LMGAIE given by

$$f(x) = \lambda \int_0^x \frac{1}{[g(x) - g(t)]^{\rho}} \varphi(t)dt, \quad 0 < \rho < 1 \tag{5}$$

In this section, the LMGAIE (5) can be solved by the Bernstein polynomials. To do so, we approximate the solution of equation (6) as a linear combination of some Bernstein polynomials as,

$$\varphi(x) \simeq \varphi_{app}(x) = \sum_{i=0}^{n} a_i\, B_{i,n}(x), \tag{6}$$

To determine the unknown coefficients a_i, we substitute equation (6) into equation (5), to obtain

$$f(x) = \lambda \int_a^x \left[\frac{1}{[g(x) - g(t)]^{\rho}} \sum_{i=0}^{n} a_i B_{i,n}(t)\right] dt, x \in [a, b] \text{ and } 0 < \rho < 1.$$

Thus, one has

$$f(x) = \sum_{i=0}^{n} a_i \left[\lambda \int_a^x \frac{1}{[g(x) - g(t)]^{\rho}} B_{i,n}(t)dt\right]$$

Now Multiplying both sides by $B_{j,n}(x)(j = 0,1,\dots,n)$ and integrating both sides concerning x from a to b, to obtain

$$\int_a^b f(x)B_{j,n}(x)dx = \sum_{i=0}^{n} a_i \int_a^b \left[\lambda \int_a^x \frac{1}{[g(x) - g(t)]^{\rho}} B_{i,n}(t)dt\right] B_{j,n}(x)dx \tag{7}$$

For simplicity, we use the following notations:

$$Q_{ij} = \int_a^b \left[\lambda \int_a^x \frac{1}{[g(x) - g(t)]^{\rho}} B_{i,n}(t)dt\right] B_{j,n}(x)dx,$$

and

$$f_j = \int_a^b f(x)B_{j,n}(x)dx.$$

Then equations (7) can be simply rewritten as,

$$\sum_{i=0}^{n} a_i Q_{ij} = f_j, \qquad j = 0,1,\dots,n. \tag{8}$$

Therefore, the linear system (8) with $(n+1)$ unknown coefficients a_i can be easily solved to numerically obtain the approximate solution $\varphi_{app}(x)$ of the LMGAIE (5).

Convergence Analysis of the Numerical Solution of LMGAIE

In this section we mention some important theorems related the existence and the convergence of the numerical solution of LMGAIE (5).

Theorem 1. (Convergence Theorem) [10] Let the functions $f(x)$ and $K(x, t)$ respectively defined on the interval $[a, b]$ and $[a, b] \times [0, t]$. Respectively, let $\psi_{exa}(x)$ and $\psi_{app}(x)$ be the exact and the approximate solution obtained by the

Bernstein polynomials of the LMGAIE (5), then

$$\lim_{n\to\infty}\|\psi_{exa}(x)-\psi_{app}(x)\|=0.$$

Proof. We start by definition of the norm as,

$$\|\psi_{exa}(x)-\psi_{app}(x)\|=\max_{\substack{x\in[a,b]\\ t\in[0,t]}}|\psi_{exa}(x)-\psi_{app}(x)|$$

$$=\max_{\substack{x\in[a,b]\\ t\in[0,t]}}\left|f(x)+\lambda\int_a^b K(x,t)\psi_{exa}(t)dt-\left[f(x)+\lambda\int_a^b K(x,t)\psi_{app}(t)dt\right]\right|$$

$$=\max_{\substack{x\in[a,b]\\ t\in[0,t]}}\left|\lambda\int_a^b K(x,t)\psi_{exa}(t)dt-\left[\lambda\int_a^b K(x,t)\psi_{app}(t)dt\right]\right|$$

$$=\max_{\substack{x\in[a,b]\\ t\in[0,t]}}\left|\lambda\int_a^b K(x,t)[\psi_{exa}(t)-\psi_{app}(t)]dt\right|$$

Now we recall the Cauchy -Schwarz inequality, to obtain

$$\le|\lambda|\max_{\substack{x\in[a,b]\\ t\in[0,t]}}\int_a^b|K(x,t)||\psi_{exa}(t)-\psi_{app}(t)|dt$$

If we suppose $Y=\max_{\substack{x\in[a,b]\\ t\in[0,t]}}|K(x,t)|$, then we get

$$\|\psi_{exa}(x)-\psi_{app}(x)\|\le|\lambda|Y\max_{\substack{x\in[a,b]\\ t\in[0,t]}}\int_a^b|K(x,t)|\,|\psi_{exa}(t)-\psi_{app}(t)|\,dt$$

Since $a\le x\le b$, then

$$\|\psi_{exa}(x)-\psi_{app}(x)\|\le\lambda Y(b-a)\|\psi_{exa}(x)-\psi_{app}(x)\|_\infty.$$

We suppose $\alpha=\lambda Y\beta$, where $\beta=(b-a)$.

$$(1-\alpha)\|\psi_{exa}(x)-\psi_{app}(x)\|_\infty\le0$$

Thus for $0\le\alpha\le1$ and $n\to\infty$,one has

$$\lim_{n\to\infty}\|\psi_{exa}(x)-\psi_{app}(x)\|=0.$$

Thus, the numerical solution obtained by the Bernstein polynomials converges to the exact solution of the LMGAIE (5).

Theorem 2. [11] Suppose that $\mathcal{H}=l^2[0,1]$ is a Hilbert space with the inner product defined by $\langle f,g\rangle=\int_0^1 f(x)\,g(x)dx$ and $\{B_{0,n},B_{1,n},\dots,B_{n,n}\}\subset\mathcal{H}$ be the set of Bernstein polynomials of n^{th} degree. Let $S_n=\text{span}\{B_{0,n},B_{1,n},\dots,B_{n,n}\}$ and f be an arbitrary element in $\mathcal{H}$. Since S_n is a finite dimensional and closed subspace, then S_n is a complete subset of $\mathcal{H}$. So f has the unique best approximation out of S_n such as $s_0\in S_n$.

Therefore, there exist unique coefficients $a_i, i=0,1,\dots,n$ such that,

$$f(x)\approx s_0(x)=\sum_{i=0}^n a_iB_{i,n}(x)=a^T\Phi_n(x),$$

where $a^T=[a_0,a_1,\dots,a_n]$ can be obtained by

$$a^T\langle\Phi_n(x),\Phi_n(x)\rangle=\langle f,\Phi_n(x)\rangle.$$

Such as,

$$\langle f,\Phi_n(x)\rangle=\int_0^1 f(x)\,\Phi_n(x)^Tdx=[\langle f,B_{0,n}\rangle,\langle f,B_{1,n}\rangle,\dots,\langle f,B_{n,n}\rangle]$$

Now let $\Theta=\langle\Phi_n(x),\Phi_n(x)\rangle$ that is a $(n+1)(n+1)$ square matrix and itis elements can be computed as,

$$\Theta_{i+1,j+1}=\int_0^1 B_{i,n}(x)B_{j,n}(x)dx, i,j=0,1,\dots n$$

Recalling the integral of Bernstein polynomials given by equation (4) leads to

$$\Theta_{i+1,j+1}=\frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}}.$$

Lemma 1. [11] Let $f:[0,1]\to\mathcal{R}$ is $(n+1)$ times continuously differentiable, and $S_n=span\{B_{0,n},B_{1,n},\dots,B_{n,n}\}$. If $a^TB_{i,n}(x)$ is the best approximation f out of S then

$$\|f-a^TB_{i,n}(x)\|\le\frac{\max_{x\in[0,1]}|f^{(n+1)}(x)|}{(n+1)!\sqrt{2n+3}}$$

Proof. Since the set $\{1,x,\dots,x^n\}$ spans the space polynomials of degree n, thus one has

$$P_1(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\dots+\frac{x^n}{n!}f^{(n)}(0).$$

Thus, from Taylor expansion we have

$$|f(x)-P_1(x)|=\left|f^{n+1}(\xi_x)\frac{x^{n+1}}{n+1}\right|,$$

where $\xi_x\in[0,1]$. Since $a^TB_{i,n}(x)$ is the best approximation f out of S, $P_1\in S_n$, we have

$$\|f-a^TB_{i,n}(x)\|^2_{l^2[0,1]}\le\|f-P_1\|^2_{l^2[0,1]}=$$

$$=\int_0^1|f(x)-P_1(x)|^2dx$$

$$=\int_0^1|f^{n+1}(\xi_x)|^2\left(\frac{x^{n+1}}{n+1}\right)^2dx$$

$$\le\frac{\left(\max_{x\in[0,1]}|f^{(n+1)}(x)|\right)^2}{(n+1)^2!}\int_0^1x^{2n+2}dx\le\frac{\left(\max_{x\in[0,1]}|f^{(n+1)}(x)|\right)^2}{(n+1)^2!\,(2n+3)}$$

Thus, the proof is complete.

Results and Discussion

Herer, some numerical examples of the LMGAIE of both the first and second kinds are shown. The accuracy is presented in terms of the absolute error E defined as,

$$E=|\varphi_{exa}(x)-\varphi_{app}(x)|.$$

Example 1. Consider the first-kind LGAIE [12]:

$$\int_0^x\frac{1}{(x-t)^{1/3}}\varphi(t)dt=x^{5/3},\qquad x\in[0,1]\qquad(9)$$

where the exact analytic solution is $\varphi_{exa}(x)=\frac{10}{9}x$.

Solution: We constitute the approximate solution of equation (9) as,

$$\varphi_{app}(x)=\sum_{i=0}^4a_i\,B_{i,4}(x)\qquad(10)$$

Substituting equation (10) into equation (9), leads to

$$\sum_{i=0}^4a_i\left[\int_0^x\frac{1}{(x-t)^{1/3}}B_{i,4}(t)dt\right]=x^{5/3}$$

Now Multiply this equation by the functions $B_{j,n}(x)(j=0,1,2,3,4)$ and then integrate it with respect to x from 0 to 1, to obtain

$$\sum_{i=0}^4a_i\int_0^1\left[\int_0^x\frac{1}{(x-t)^{1/3}}B_{i,4}(t)dt\right]B_{j,4}(x)dx=\int_0^1x^{5/3}B_{j,4}(x)dx$$

Now, we suppose that:

$$W_{ij}=\int_0^1\left[\int_0^x\frac{1}{(x-t)^{1/3}}B_{i,4}(t)dt\right]B_{j,4}(x)dx,\qquad i,j=0,1,2,3,4$$

And: $f_j=\int_0^1x^{5/3}B_{j,4}(x)dx\,,\ j=0,1,2,3,4$

Thus, one has the following short form,

$$\sum_{i=0}^{n} a_i W_{ij} = f_j$$

Thus, the LGAIE (9) is converted to a linear system of algebraic equation of the unknowns a_0, a_1, a_2, a_3, a_4 which can be easily solved to obtain the numerical solution as,

$$\varphi_{app}(x) = -1.469 \times 10^{-15} + 1.111x - 1.421 \times 10^{-13}x^2 + 2.411 \times 10^{-13}x^3 - 1.306 \times 10^{-13}x^4$$

Table 1 and Fig. 1 present the exact and the approximate solutions with the absolute error E calculated at some points within the given domain $[0,1]$. The obtained results confirm the efficiency of the method even for small degree of the Bernstein polynomials $n = 4$.

Table 1: Numerical results of the LGAIE (9) for $n = 4$

x	$\varphi_{exa}(x)$	$\varphi_{app}(x)$	E
0	0	$-1.4 * 10^{-15}$	$1.4697279217 * 10^{-15}$
0.1	0.111111111	0.111111111	$2.0816681711 * 10^{-16}$
0.2	0.222222222	0.222222222	$3.0531133177 * 10^{-16}$
0.3	0.333333333	0.333333333	$1.6653345369 * 10^{-16}$
0.4	0.444444444	0.444444444	$6.1062266354 * 10^{-16}$
0.5	0.555555555	0.555555555	$6.6613381477 * 10^{-16}$
0.6	0.666666666	0.666666666	$2.2204460492 * 10^{-16}$
0.7	0.777777777	0.777777777	$3.3306690738 * 10^{-16}$
0.8	0.888888888	0.888888888	$5.5511151231 * 10^{-16}$
0.9	1	0.999999999	$7.7715611723 * 10^{-16}$
1	1.111111111	1.111111111	$4.4408920985 * 10^{-16}$

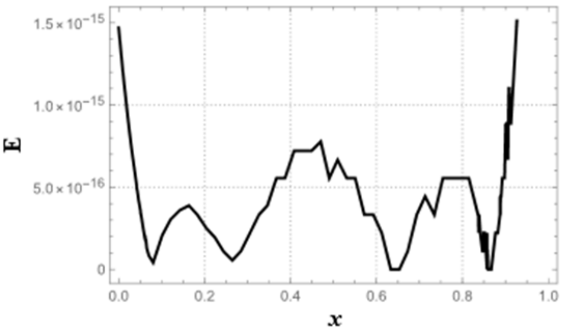

Figure 1: Absolute error for $n = 4$ of equation (9)

Example 2. Consider the following first-kind LMGAIE [1]:

$$\int_0^x \frac{1}{(x^5 - t^5)^{\frac{1}{6}}} \varphi(t)dt = \frac{6}{25} x^{25/6}, \qquad x \in (0,2) \qquad (11)$$

where the analytical solution is $\varphi_{ext}(x) = x^4$.

Solution: Similarly to example 1, we implement the Bernstein polynomials with degree $n = 6$ to obtain the numerical solution as,

$$\varphi_{app}(x) = 3.508 \times 10^{-15} - 8.549 \times 10^{-14}x + 7.175 \times 10^{-13}x^2 - 1.632 \times 10^{-12}x^3 + x^4 - 7.9186 \times 10^{-13}x^5 + 1.425 \times 10^{-13}x^6$$

Table 2 depicts the exact and the approximate solutions with the absolute error E calculated at some chosen points within the domain $(0,2)$.

Example 3: Consider the first-kind LMGAIE [1]:

$$\int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{2}}} \varphi(t)dt = \frac{2}{3}\pi x^3, \qquad x \in (0,2) \qquad (12)$$

where the analytical solution is $\varphi_{ext}(x) = \pi x^3$.

Table 2. Numerical results of equation (11) for $n = 6$

x	$\varphi_{exa}(x)$	$\varphi_{app}(x)$	E
0	0	$3.508 * 10^{-15}$	$3.5088709018 * 10^{-15}$
0.2	0.001600000	0.0016000004	$4.4677803123 * 10^{-15}$
0.4	0.0256000000	0.0256000014	$1.4672291159 * 10^{-14}$
0.600001	0.1296000008	0.1296000018	$1.8429702208 * 10^{-14}$
0.8	0.4096000002	0.4096000017	$1.7319479184 * 10^{-14}$
1	1	1.0000000016	$1.6209256159 * 10^{-14}$
1.2	2.0736	2.0736000015	$1.5543122344 * 10^{-14}$
1.4	3.8415999999	3.841600001`	$1.1102230246 * 10^{-14}$
1.599999	6.5535999998	6.5535999998	$8.8817841970 * 10^{-16}$
1.799999	10.497599999	10.497599998	$1.0658141036 * 10^{-14}$
1.999999	15.999999999	16.000000001	$2.1316282072 * 10^{-14}$

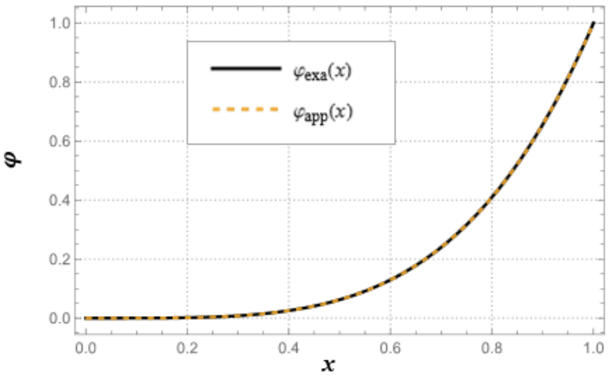

Fig. 2: Exact and approximate solutions of equation (11)

Example 3: Consider the first-kind LMGAIE [1]:

$$\int_0^x \frac{1}{(x^2 - t^2)^{\frac{1}{2}}} \varphi(t)dt = \frac{2}{3}\pi x^3, \qquad x \in (0,2) \qquad (12)$$

where the analytical solution is $\varphi_{ext}(x) = \pi x^3$.

Solution: In a similar fashion to above, we apply the Bernstein polynomials with degree $n = 5$ to obtain the approximate solution as,

$$\varphi_{app}(x) = -5.834 \times 10^{-16} + 1.793 \times 10^{-14}x - 1.963 \times 10^{-13}x^2 + 3.142x^3 - 9.560 \times 10^{-13}x^4 + 4.448 \times 10^{-13}x^5$$

Table 3 depicts the exact and the approximate solutions with the absolute error E computes at some points within the domain $(0,2)$.

Example 4: Solve the second-kind Abel's integral equation [13]:

$$4\varphi(x) = \frac{4}{\sqrt{x+1}} - \arcsin\left(\frac{1-x}{1+x}\right) + \frac{\pi}{2} - \int_0^x \frac{1}{\sqrt{x-t}} \varphi(t)dt \qquad (13)$$

where $0 \le x < 1$ and the exact solution is given as, $\varphi_{exa}(x) = \frac{1}{\sqrt{x+1}}$.

Solution: To check the convergence of the considered method, we compute the numerical solution of equation (13) for three different values of n, that is for $n = 3, 7, 9$ by the same steps as above.

Table 3: Numerical results of equation (12) for $n = 5$

x	$\varphi_{exa}(x)$	$\varphi_{app}(x)$	E
0.2	0.0251327412	0.0251327412	$5.8336426184 * 10^{-16}$
0.4	0.2010619298	0.2010619298	$6.6613381477 * 10^{-16}$
0.600000	0.6785840131	0.6785840131	$1.9428902930 * 10^{-16}$
0.8	1.6084954386	1.6084954386	$5.5511151231 * 10^{-16}$
1	3.1415926535	3.1415926535	$1.3322676295 * 10^{-15}$
1.2	5.4286721054	5.4286721054	$5.7731597280 * 10^{-15}$
1.4	8.6205302414	8.6205302414	$6.4837024638 * 10^{-14}$
1.599999	12.867963509	12.867963509	$2.7000623958 * 10^{-13}$
1.799999	18.321768355	18.321768355	$7.7626793881 * 10^{-13}$
1.999999	25.132741228	25.132741228	$1.8225421172 * 10^{-12}$

Table 4: The absolute errors for equation (13) for $n = 3, 7, 9$.

x	E for $n = 3$	E for $n = 7$	E for $n = 9$
0	$8.3950 * 10^{-4}$	$7.326397 * 10^{-7}$	$2.04974 * 10^{-8}$
0.1	$2.2593 * 10^{-4}$	$1.996567 * 10^{-8}$	$6.22674 * 10^{-9}$
0.2	$2.9543 * 10^{-4}$	$1.157989 * 10^{-7}$	$5.16797 * 10^{-9}$
0.3000	$2.9869 * 10^{-5}$	$1.821915 * 10^{-7}$	$2.86788 * 10^{-9}$
0.4	$2.0729 * 10^{-4}$	$9.029815 * 10^{-9}$	$1.37488 * 10^{-9}$
0.5	$2.5913 * 10^{-4}$	$1.641267 * 10^{-7}$	$4.38498 * 10^{-9}$
0.6	$1.1621 * 10^{-4}$	$5.429183 * 10^{-8}$	$3.15477 * 10^{-9}$
0.7	$1.2309 * 10^{-4}$	$1.430505 * 10^{-7}$	$7.7701 * 10^{-10}$
0.7999	$2.7998 * 10^{-4}$	$1.404579 * 10^{-7}$	$3.78437 * 10^{-9}$
0.8999	$1.1448 * 10^{-4}$	$3.461212 * 10^{-8}$	$4.91926 * 10^{-9}$
0.9999	$6.6056 * 10^{-4}$	$5.677707 * 10^{-7}$	$1.74899 * 10^{-8}$

Table 4 depicts the absolute error E calculated at some points within the interval $[0,1)$ for $n = 3, 7, 9$ and clearly confirms the convergence of the method where we achieve very good accuracy for small degree of the Bernstein polynomials.

Conclusions

In this paper, we implemented the Bernstein polynomials for solving various linear integral equations with weakly-singular kernels, namely Abel's, generalized Abel's and main generalized Abel's integral equations. Simulations were performed using Mathematica 14.1 (Wolfram Research, Inc.) [14] A comparison is carried out between the numerical approximate solution against the given analytic solutions. In summary, the numerical findings obtained from this analysis provide compelling evidence that supports the effectiveness and dependability of the Bernstein polynomials method. These results not only validate its applications but also reinforce its standing as a reliable mathematical tool for solving relevant problems. The approximate solution converged to the exact solution for small values of the Bernstein polynomials degree (n) as shown in Tables 1, 2, 3, 4.

Author Contributions: "**Gharh** and **Saed**: Conceptualization and methodology, writing—original draft preparation, review and editing. **Gharh** and **Saed**: results' analysis and discussion. Both authors have read and agreed to the published version of the manuscript."
Funding: "This research received no external funding."
Data Availability Statement: "No data were used to support this study."
Conflicts of Interest: "The authors declare that they have no conflict of interest."

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