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RESEARCH ARTICLE **FLUID MECHANICS**

Higher-Order Recurrent Finsler Geometry and Anisotropic Flow Modelling with CFD Applications

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ABSTRACT

In this work, a unified high-level framework that links anisotropic flow via higher-order recurrence structures with Finsler geometry is developed. It focuses on the connections between Weyl projective and concircular curvature tensors in extended second-order recurrent and birecurrent Finsler spaces. By obtaining new tensorial relations and strict necessary and sufficient conditions, classical recurrence theory is expanded. To confirm applicability, CFD simulations of viscous flow in curved pipes with varying curvatures and intake conditions are performed. The results demonstrate significant directional dependency in velocity and wall shear stress. The results are consistent with the proposed geometric model, indicating a close relationship between advanced differential geometry and anisotropic fluid behavior with implications for modified gravity and cosmology.

الهندسة الفينسلرية التكرارية من الرتب العليا ونمذجة الجريان اللاتماثلي باستخدام تطبيقات ديناميكا الموائع الحسابية (CFD)

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الملخص

في هذا العمل، تم تطوير إطار موحّد عالي المستوى يربط بين الجريان اللاتماثلي وبنى التكرارية من الرتب العليا وهندسة فينسلر. ويركز الإطار على العلاقات بين موتر الانحناء الإسقاطي لوايل (Weyl projective curvature tensor) وموتر الانحناء الدائري (concircular curvature tensor) في فضاءات فينسلر التكرارية الممتدة من الرتبة الثانية والفضاءات الفينسلرية ثنائية التكرار (birecurrent Finsler spaces). ومن خلال اشتقاق علاقات موترية جديدة وشروط ضرورية وكافية صارمة، جرى توسيع نظرية التكرار الكلاسيكية. وللتحقق من قابلية التطبيق، أُجريت محاكاة باستخدام ديناميكا الموائع الحسابية (CFD) لجريان لزج داخل أنابيب منحنية ذات انحناءات متفاوتة وظروف دخول مختلفة. وأظهرت النتائج اعتمادًا اتجاهيًا ملحوظًا لكلٍّ من السرعة وإجهاد القص على الجدار. كما تتوافق النتائج مع النموذج الهندسي المقترح، مما يشير إلى وجود علاقة وثيقة بين الهندسة التفاضلية المتقدمة والسلوك اللاتماثلي للموائع، مع ما قد يترتب على ذلك من تطبيقات في نماذج الجاذبية المعدّلة وعلم الكونيات.

Introduction

In both mathematics and science, Finsler geometry a logical extension of Riemannian geometry offers a strong and adaptable foundation for describing anisotropic structures. Finsler spaces are especially useful for characterizing systems with intrinsic directional dependency because, in contrast to normal Riemannian spaces, they permit the metric to depend on both location and direction. Because of its applications in differential geometry, gravitational theories, and cosmology, this geometric framework has gained more and more attention in recent years. The study of recurrent structures, which characterize the behavior of curvature tensors under

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covariant differentiation, is one of the main topics of Finsler geometry. These structures have been thoroughly studied in numerous contexts and are essential to comprehending the intrinsic geometry of manifolds. In particular, higher-order recurrence has grown to be a substantial extension of classical recurrence theory, offering a more profound comprehension of the behavior and connections of curvature tensors. Birecurrent and generalized second-order recurrent spaces fall within this category.

The Weyl projective curvature tensor and the concircular curvature tensor are two important geometric objects in this setting. Important details regarding the underlying space's geometric and physical characteristics are encoded by these tensors. The structural interactions between them under higher-order recurrence conditions have not been thoroughly examined, despite the fact that their individual features have. To create a more thorough understanding of anisotropic geometric structures, it is essential to comprehend these interactions.

Connections between abstract geometric frameworks and real-world applications are becoming more and more common alongside these theoretical advancements. The ramifications of direction-dependent geometric structures can be naturally illustrated in the context of anisotropic behavior in fluid dynamics. For the analysis of intricate flow processes in geometries where curvature and directional effects are important, like flow through curved pipes, computational fluid dynamics (CFD) has emerged as a potent tool.

Inspired by these ideas, the goal of this work is to develop a logical framework that connects anisotropic flow behavior with higher-order recurrence in Finsler geometry. We theoretically study the structural relationships between Weyl projective and concircular curvature tensors in extended second-order recurrent and birecurrent Finsler spaces. In addition to establishing necessary and sufficient conditions that expand standard results in recurrence theory, we construct novel tensorial relations. In practical terms, we use CFD models to study viscous flow in curved pipe designs with different intake velocities and curvature radii. The numerical results provide a physical explanation that is consistent with the geometric framework and clearly show anisotropic aspects in the velocity distribution and wall shear stress. The importance of higher-order geometric structures in simulating actual transport phenomena is shown by this dual approach. The development and use of a higher-order recurrence framework in Finsler geometry, the discovery of novel connections between Weyl and concircular curvature tensors, and the fusion of geometric theory with CFD-based flow analysis are the three primary contributions of this work. With possible ramifications for cosmology and modified theories of gravity, this combination offers a unique link between applied fluid dynamics and abstract differential geometry.

This is how the rest of the paper is structured. Background information and the essential terminology are included in Section 2. The theoretical findings on curvature tensor interactions are developed in Section 3. The numerical CFD research and its geometric interpretation are presented in Section 4. Conclusions and possible future paths are finally covered. In this direction, generalized recurrent Finsler spaces of higher order associated with Berwald curvature have been studied, providing structural relations between higher-order covariant derivatives and curvature tensors [1]. Further developments considered generalized (K)-recurrent conditions in Finsler spaces, thereby extending recurrent geometric structures and contributing to a broader understanding of curvature-dependent differential relations [2]. Higher-order differential operators and curvature decompositions constitute another important aspect of modern Finsler geometry. The systematic investigation of higher-order Cartan derivatives and curvature tensor decompositions have demonstrated the relevance of these structures to both mathematical and physical applications [3]. Related studies have also examined the (M)-projective curvature tensor in generalized recurrent and birecurrent Finsler spaces, highlighting the role of curvature transformations in describing generalized geometric configurations [4]. These investigations provide a useful theoretical background for the present work because they emphasize the importance of higher-order differential structures, generalized recurrence, and nonlinear geometric relations in the mathematical description of complex systems.

Recent advances in Finsler geometry have further expanded the study of generalized curvature and anisotropic geometric structures. Finsler metrics involving generalized Ricci tensors have been investigated to establish broader classes of curvature conditions and geometric characteristics [5], while the study of Finsler metrics with vanishing mean Cartan torsion has contributed to the understanding of special metric structures and their associated geometric constraints [6]. Pseudo-Finsler metrics on radially symmetric spaces have also been considered, providing additional insight into the geometric behavior of anisotropic spaces under symmetry assumptions [7]. Moreover, warped-product Finsler metrics with special curvature properties have been developed as another important class of generalized geometric models [8].

The interaction between Finsler geometry and modern mathematical physics has received increasing attention. Studies on recurrent Weyl manifolds have examined generalized curvature structures and their geometric properties [9], whereas the geometry of Finslerian connections has been explored in connection with physical applications [10]. More recent investigations have considered Finslerian wormhole geometries in modified gravity [11] and higher-spin interactions within Finsler-type frameworks [12]. The relevance of Finsler geometry to cosmological modeling has likewise been discussed, demonstrating the potential of anisotropic geometric formulations in describing large-scale physical systems [13]. Generalized Berwald metrics and their curvature properties provide another significant direction in this context [14], while generalized projective Finsler spaces have been investigated in relation to broader geometric and applied structures [15]. Reviews and theoretical studies have additionally emphasized the potential role of Finsler geometry in cosmology and modified gravitational theories [16,17].

In parallel with these developments, generalized recurrent Finsler spaces continue to represent an active area of research, particularly in relation to curvature tensors and higher-order differential conditions [18]. Collectively, these studies demonstrate the continuing importance of generalized differential and geometric structures in modeling nonlinear and anisotropic phenomena. Nevertheless, the aforementioned literature is predominantly concerned with geometric structures, curvature tensors, recurrence conditions, and their physical interpretations, rather than with the derivation of approximate analytical solutions for time-fractional nonlinear evolution equations. As well as the use of

CFD in other applications [19,20].

This observation motivates the present study, which focuses on the time-fractional Chen-Lee-Liu equation and employs the Asymptotic Homotopy Perturbation Method (AHPM) to construct approximate analytical solutions. The method combines the flexibility of homotopy-based techniques with asymptotic perturbation concepts, offering an effective analytical framework for treating nonlinear fractional differential equations without requiring conventional linearization or restrictive assumptions. In contrast to the geometric studies summarized above, the present work emphasizes the analytical treatment of a nonlinear time-fractional evolution equation and the construction of approximate solution expressions that can capture the influence of the fractional temporal derivative. The study therefore contributes to the broader intersection of nonlinear analysis, fractional calculus, and mathematical modeling by providing an analytical approach for investigating the dynamics of the time-fractional Chen-Lee-Liu equation.

Finally, the numerical visualization based on CFD offers a useful viewpoint on the geometric framework created in this study. It shows that directional flow behavior, velocity asymmetry, and wall shear stress are all greatly influenced by curvature and geometric limitations. These results provide a scientific explanation of anisotropic properties similar to generalized second-order recurrent structures in Finsler geometry. The conceptual relevance of higher-order recurrence in modeling transport processes is supported by the dependency on direction and curvature, despite the lack of a direct equivalency. The usefulness of such geometric structures in fluid mechanics and engineering systems is thus highlighted in this paper. Additionally, it implies their possible applicability in scientific and cosmological models where curvature effects and intrinsic anisotropy are essential.

Preliminaries

A few definitions and criteria will be provided in this section for the sake of this study.

The following criteria are met by two vectors, y_i and y^i.

a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$, and

b) c) $\delta_j^k y^j = y^k$. (1)

The relationship between the quantities g_{ij} and g^{ij} is

a) $g_{ij}\, g^{jk} = \delta_i^k = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \end{cases}$

b) $g_{ir}\, \delta_j^i = g_{rj}$ and c) $g^{jk} = \delta_k^i = g^{ji}$. (2)

For any vector field, the h-covariant derivative of second order with regard to x^k and x^j is as follows:

$X^i_{|k|j} = \partial_j\big(X^i_{|k}\big) - \big(X^i_{|r}\big)\Gamma^{*r}_{kj} + \big(X^r_{|k}\big)\Gamma^{*i}_{rj} - \big(\partial_j X^i_{|k}\big)\Gamma^{*i}_{js} y^s$. (3)

Under Cartan's covariant derivative, the vector y^i and metric function F vanish in the same way.

a) $F_{|h} = 0$ and b) $y^i{}_{|h} = 0$. (4)

The definition of the metric tensors g_{ij} and g^{ij} vanish identically under Cartan's covariant derivative

a) $g_{ij|h} = 0$ and b) $g^{ij}{}_{|h} = 0$. (5)

Tensor W^i_{jkh} , torsion tensor W^i_{jk} and deviation tensor W^i_j are defined by:

$$W^i_{jkh} = H^i_{jkh} + \frac{2\,\delta^i_j}{(n+1)} H_{[hk]} + \frac{2\,y^i}{(n+1)} \dot\partial_j H_{[kh]} + \frac{\delta^i_k}{(n^2-1)}\big(n\, H_{jh} + H_{hj} + y^r \dot\partial_j H_{hr}\big) - \frac{\delta^i_h}{(n^2-1)}\big(n\, H_{jk} + H_{kj} + y^r \dot\partial_j H_{kr}\big) \ , \quad (6)$$

$$W^i_{jk} = H^i_{jk} + \frac{y^i}{(n+1)} H_{[jk]} + 2\{\frac{\delta^i_{[j}}{(n^2-1)}\big(n\, H_{k]} - y^r H_{k]\,r}\big)\} \ . \quad (7)$$

And $W^i_j = H^i_j - H\,\delta^i_j - \frac{1}{(n+1)}\big(\dot\partial_r H^r_j - \dot\partial_j H\big)\, y^i$, (8)

respectively.

Also, if we suppose that the tensor W^i_j satisfy the following identities

a) $W^i_k\, y^k = 0$, b) $W^i_i = 0$, c) $W^i_k\, y_i = 0$.

d) $g_{ir}\, W^i_j = W_{rj}$, e) $g^{jk} W_{jk} = W$, and

f) $W_{jk}\, y^k = 0$. (9)

The concircular curvature tensor M^i_{jkh} , torsion tensor M^i_{jk}, Ricci tensor M_{jk}, curvature vector M_k and scalar curvature M are satisfying:

a) $M^i_{jkh}\, y^j = M^i_{kh}$, b) $M^i_{kh}\, y^k = M^i_h$,

c) $M^i_{jki} = M_{jk}$, d) $M^i_{ki} = M_k$, e) $M^i_i = M$, and

f) $g_{ir}\, M^i_{jkh} = M_{rjkh}$. (10)

Cartan's 3th curvature tensor R^i_{jkh} , Ricci tensor R_{jk} , the vector H_k and scalar curvature H are defined as

a) $R_{jk}\, y^j = H_k$, b) $R_{jk}\, y^k = R_j$, c) $R^i_i = R$, and

d) $H_k\, y^k = (n-1)H$. (11)

Using Cartan's and Berwald's first and second order derivatives in Finsler space, which are defined by the following criteria, Al-Qashbari and AL-ssallal [6] and Al-Qashbari, Haouse, and AL-ssallal [7] developed and investigated generalization of Weyl's projective curvature tensor in Finsler spaces.

$$W^i_{jkh|m} = \lambda_m W^i_{jkh} + \mu_m\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) + \frac{1}{4}\gamma_m\big(W^i_k g_{jh} - W^i_h g_{jk}\big) \ . \quad (12)$$

The generalized $W_{|h}$-recurrent space, denoted by $\mathrm{G^{2nd}}\,W_{|h}$-$\mathrm{RF_n}$, is a Finsler space $\mathrm{F_n}$ in which the curvature tensor W^i_{jkh} meets the condition (12).

Applying the constraint (5a) to the covariant derivative of (12) with regard to x^l in the context of Cartan's connection yields:

$$W^i_{jkh|m|l} = a_{ml} W^i_{jkh} + b_{ml}\big(\delta^i_k\, g_{jh} - \delta^i_h\, g_{jk}\big) + \frac{1}{4} c_{ml}\big(W^i_k g_{jh} - W^i_h g_{jk}\big) + \frac{1}{4}\gamma_m\big(W^i_k\, g_{jh} - W^i_h\, g_{jk}\big)_{|l} \ . \quad (13)$$

where $a_{ml} = \lambda_{m|l} + \lambda_m \lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m \mu_l$ and $c_{ml} = \lambda_m \gamma_l + \gamma_{m|l}$ are non-zero second order covariant tensor fields, γ_m is non-zero first order covariant vector field, respectively.

Definition 1. In Finsler space, which the Wely's projective curvature tensor W^i_{jkh} satisfies the condition (13) is called the generalization generalized $W_{|h}$ -birecurrent space and the tensor will be called a generalization generalized h -birecurrent space. These space and tensor denote them briefly by $G^{2nd}W_{|h}$- BRF_n and $G^{2nd}h$-BR, respectively. We consider an n-dimensional Finsler space F_n , the Weyls projective curvature tensor W^i_{jkh} satisfies the condition (12) and (13), These spaces denoted by $G^{2nd}W_{|h}$-RF_n and $G^{2nd}W_{|h}$- BRF_n , respectively.

Relationship Between Wely's Curvature Tensor W^i_{jkh} and Concircular Curvature Tensor M^i_{jkh}

The concircular curvature tensor is a geometric object introduced in differential geometry. It is a measure of the curvature of spacetime or, more generally, a pseudo-Riemannian manifold. Finsler geometry, as a generalization of Riemannian geometry, provides a powerful framework for modeling a wide range of physical phenomena. In Finsler spaces, the curvature properties of the space are characterized by various curvature tensors, among which Weyl and the concircular curvature tensor M^i_{jkh} play a significant role. While the geometric interpretations and physical implications of these tensors have been extensively studied, the relationship between them remains a subject of ongoing research.

This paper aims to investigate the connection between Weyl's curvature tensor and the concircular curvature tensor M^i_{jkh} in Finsler spaces. By exploring their algebraic and geometric properties, we seek to establish new identities and inequalities that relate these two tensors. Our findings are expected to contribute to a deeper understanding of the curvature structure of Finsler spaces and provide insights into their applications in physics, such as in the study of gravitational theories and cosmology. Some properties of W^i_{jkh} curvature tensor was proposed by Al-Qashbari, Abdallah and Al-ssallal.

For $(n = 4)$ a Riemannian space, Weyl defined the concircular curvature tensor M^i_{jkh} often known as the Weyl concircular tensor, as

$$W^i_{jkh} = M^i_{jkh} + \tfrac{1}{12}R\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \tag{14}$$

Taking the covariant derivative of (14), with respect to x^m in the context of Berwald we get

$$W^i_{jkh|m} = M^i_{jkh|m} + \tfrac{1}{12}\left[R\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right)\right]_{|m} + \tfrac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m} . \tag{15}$$

Using (5a), in the equation (15) can be written as

$$W^i_{jkh|m} = M^i_{jkh|m} + \tfrac{1}{12}R_{|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{3}\left(\delta^i_k R_{jh|m} - g_{jk} R^i_{h|m}\right) . \tag{16}$$

By substituting equations (12) and (14) in to (16), we obtain:

$$\begin{aligned} M^i_{jkh|m} &+ \tfrac{1}{12}R_{|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{3}\left(\delta^i_k R_{jh|m} - g_{jk} R^i_{h|m}\right) \\ &= \lambda_m M^i_{jkh} + \tfrac{1}{12}\lambda_m R\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &\quad + \tfrac{1}{3}\lambda_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &\quad + \tfrac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right) . \end{aligned} \tag{17}$$

The equation (17), can be expressed as:

$$\begin{aligned} M^i_{jkh|m} &= \lambda_m M^i_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &\quad + \tfrac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right) - \tfrac{1}{12}R_{|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &\quad - \tfrac{1}{3}\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m} + \tfrac{1}{12}\lambda_m R\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) \\ &\quad + \tfrac{1}{3}\lambda_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \end{aligned} \tag{18}$$

This demonstrates that

$$M^i_{jkh|m} = \lambda_m M^i_{jkh} + \mu_m\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W^i_k g_{jh} - W^i_h g_{jk}\right) . \tag{19}$$

If and only if

$$R_{|m}\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) = \lambda_m R\left(\delta^i_k g_{jh} - \delta^i_h g_{jk}\right) , \text{ and}$$
$$\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right)_{|m} = \lambda_m\left(\delta^i_k R_{jh} - g_{jk} R^i_h\right) . \tag{20}$$

Consequently, we can state that the theorem's proof is finished.

Theorem 1: If the condition (20) is met, the concircular curvature tensor M^i_{jkh} represents a generalized recurrent Finsler space in the space $G^{2nd}M_{|h}\text{-}RF_n$.

The following outcome is obtained by transvecting equation (18) using y^j and applying equations (10a), (4b), (1a), and (11a):

$$\begin{aligned} M^i_{kh|m} &= \lambda_m M^i_{kh} + \mu_m\left(\delta^i_k y_h - \delta^i_h y_k\right) \\ &\quad + \tfrac{1}{4}\gamma_m\left(W^i_k y_h - W^i_h y_k\right) - \tfrac{1}{12}R_{|m}\left(\delta^i_k y_h - \delta^i_h y_k\right) \\ &\quad - \tfrac{1}{3}\left(\delta^i_k H_h - y_k R^i_h\right)_{|m} + \tfrac{1}{3}\lambda_m\left(\delta^i_k H_h - y_k R^i_h\right) \\ &\quad + \tfrac{1}{12}\lambda_m R\left(\delta^i_k y_h - \delta^i_h y_k\right) . \end{aligned} \tag{21}$$

This demonstrates that

$$M^i_{kh|m} = \lambda_m M^i_{kh} + \mu_m\left(\delta^i_k y_h - \delta^i_h y_k\right) + \tfrac{1}{4}\gamma_m\left(W^i_k y_h - W^i_h y_k\right) . \tag{22}$$

If and only if

$$R_{|m}\left(\delta^i_k y_h - \delta^i_h y_k\right) = \lambda_m R\left(\delta^i_k y_h - \delta^i_h y_k\right) , \text{ and}$$
$$\left(\delta^i_k H_h - y_k R^i_h\right)_{|m} = \lambda_m\left(\delta^i_k H_h - y_k R^i_h\right) . \tag{23}$$

Therefore, the proof of theorem is completed, we can say

Theorem 2. In the space $G^{2nd}M_{|h}\text{-}RF_n$, the torsion tensor M^i_{kh} (concircular curvature tensor M^i_{jkh}) represents a generalized recurrent Finsler space, provided that the condition (23) is satisfied.

By transvecting (21) with y^k and applying along with equations (10b), (4a), (4b), (1b), (9a), (1c) and (11e), we obtain the following result:

$$\begin{aligned} M^i_{h|m} &= \lambda_m M^i_h + \mu_m\left(y^i y_h - \delta^i_h F^2\right) + \tfrac{1}{4}\gamma_m\left[W^i_h F^2\right] \\ &\quad - \tfrac{1}{12}R_{|m}\left(y^i y_h - \delta^i_h F^2\right) - \tfrac{1}{3}\left(y^i H_h - F^2 R^i_h\right)_{|m} \\ &\quad + \tfrac{1}{3}\lambda_m\left(y^i H_h - F^2 R^i_h\right) + \tfrac{1}{12}\lambda_m R\left(y^i y_h - \delta^i_h F^2\right) . \end{aligned} \tag{24}$$

This demonstrates that

$$M^i_{h|m} = \lambda_m M^i_h + \mu_m\left(y^i y_h - \delta^i_h F^2\right) + \tfrac{1}{4}\gamma_m\left(W^i_h F^2\right) . \tag{25}$$

If and only if

$$R_{|m}\left(y^i y_h - \delta^i_h F^2\right) = \lambda_m R\left(y^i y_h - \delta^i_h F^2\right) , \text{ and}$$
$$\left(y^i H_h - F^2 R^i_h\right)_{|m} = \lambda_m\left(y^i H_h - F^2 R^i_h\right) . \tag{26}$$

Therefore, the proof of theorem is completed, we can say

Theorem 3. In the space $G^{2nd}M_{|h}\text{-}RF_n$, the projective deviation tensor M^i_h represents a generalized recurrent Finsler space if the tensors $R\left(y^i y_h - \delta^i_h F^2\right)$ and $\left(y^i H_h - F^2 R^i_h\right)$ are generalized recurrent Finsler spaces.

By contracting the indices i and h in the equations (18), (21) and (24), respectively, and utilizing equations (n=4), (2a), (1a), (1b), (9b), (9c), (9d), (11c), (11e) and (1c), in conjunction with (10c) and (10d), we obtain the following result:

$$\begin{aligned} M_{jk|m} &= \lambda_m M_{jk} + (1-n)\mu_m g_{jk} - \tfrac{1}{4}\gamma_m W_{jk} \\ &\quad - \tfrac{1}{12}(1-n)(R g_{jk})_{|m} - \tfrac{1}{3}\left(R_{jk} - g_{jk}\mathrm{R}\right)_{|m} \\ &\quad + \tfrac{1}{3}\lambda_m\left(R_{jk} - g_{jk}\mathrm{R}\right) + \tfrac{1}{12}(1-n)\lambda_m\left(R g_{jk}\right) . \end{aligned} \tag{27}$$

This demonstrates that

$$M_{jk|m} = \lambda_m M_{jk} + (n-1)\mu_m g_{jk} - \tfrac{1}{4}\gamma_m W_{jk} . \tag{28}$$

If and only if

$$\left(R g_{jk}\right)_{|m} = \lambda_m\left(R g_{jk}\right) , \quad \text{and}$$
$$\left(R_{jk} - g_{jk}\mathrm{R}\right)_{|m} = \lambda_m\left(R_{jk} - g_{jk} R\right) . \tag{29}$$

And

$$\begin{aligned} M_{k|m} &= \lambda_m M_k + (1-n)\,\mu_m y_k - \tfrac{1}{3}(H_k - y_k\mathrm{R})_{|m} \\ &\quad - \tfrac{1}{12}(1-n)(R y_k)_{|m} + \tfrac{1}{12}(1-n)\lambda_m R y_k \\ &\quad + \tfrac{1}{3}\lambda_m(H_k - y_k\mathrm{R}) . \end{aligned} \tag{30}$$

This demonstrates that

$$M_{k|m} = \lambda_m M_k + (n-1)\mu_m\, y_k . \tag{31}$$

If and only if

$$(R y_k)_{|m} = \lambda_m(R y_k) , \text{ and}$$
$$(H_k - y_k\mathrm{R})_{|m} = \lambda_m(H_k - y_k R) . \tag{32}$$

In the last

$$M_{|m} = \lambda_m M + (1-n)\mu_m F^2 - \tfrac{1}{12}(1-n)(RF^2)_{|m} - \tfrac{1}{3}(3H - F^2\mathrm{R})_{|m} + \tfrac{1}{12}(1-n)\,\lambda_m (RF^2) + \tfrac{1}{3}\lambda_m(3H - F^2R)\ . \tag{33}$$

This demonstrates that

$$M_{|m} = \lambda_m M + (1-n)\,\mu_m F^2. \tag{34}$$

If and only if

$$(RF^2)_{|m} = \lambda_m (RF^2)\ , \text{ and } (3H - F^2\mathrm{R})_{|m} = \lambda_m(3H - F^2R)\,. \tag{35}$$

Therefore, the proof of theorem is completed, we conclude

Theorem 4. In the space $\mathrm{G^{2nd}M_{|h}\text{-}RF_n}$, Ricci tensor M_{jk} , vector M_k and scalar M are defined in equations (28), (31) and (34), respectively, if and only if the conditions in equations (29), (32) and (35) are satisfied.

By transvecting equation (18) with $g_{ir}$and utilizing equations (2b), (9d), and (10f), we obtain the following result:

$$M_{rjkh|m} = \lambda_m M_{rjkh} + \mu_m\big(g_{rk}g_{jh} - g_{rh}g_{jk}\big) + \tfrac{1}{4}\gamma_m\big[W_{rk}g_{jh} - W_{rh}g_{jk}\big] - \tfrac{1}{12}R_{|m}\big(g_{rk}g_{jh} - g_{rh}g_{jk}\big) - \tfrac{1}{3}\big(g_{rk}R_{jh} - g_{jk}\,R_{rh}\,\big)_{|m} + \tfrac{1}{12}\lambda_m R\big(g_{rk}g_{jh} - g_{rh}g_{jk}\big) + \tfrac{1}{3}\lambda_m\big(g_{rk}R_{jh} - \ g_{jk}\,R_{rh}\,\big)\ . \tag{36}$$

This demonstrates that

$$M_{rjkh|m} = \lambda_m M_{rjkh} + \mu_m\big(g_{rk}g_{jh} - g_{rh}g_{jk}\big) + \tfrac{1}{4}\gamma_m\big[W_{rk}g_{jh} - W_{rh}g_{jk}\big]\ . \tag{37}$$

If and only if

$$R_{|m}\big(g_{rk}g_{jh} - g_{rh}g_{jk}\big) = \lambda_m R\big(g_{rk}g_{jh} - g_{rh}g_{jk}\big)\ \text{, and } \big(g_{rk}R_{jh} - g_{jk}\,R_{rh}\,\big)_{|m} = \lambda_m\big(g_{rk}R_{jh} - g_{jk}\,R_{rh}\big)\,. \tag{38}$$

Therefore, we conclude that the experiment was a success

Theorem 5. In the space $\mathrm{G^{2nd}M_{|h}\text{-}RF_n}$, the associate tensor M_{jrkh} (Concircular curvature tensor M^i_{jkh}) represents a generalized recurrent Finsler space if the condition in equation (38) is satisfied.

We introduce a new class of Finsler spaces, namely, generalized $\mathrm{M_{|h}}$-birecurrent spaces. These spaces generalize the concept of birecurrence to a broader setting and exhibit interesting geometric properties. We investigate the curvature tensor of these spaces and establish several characterization theorems. Our work in this paper we defined $|m|l$ is covariant derivative of second order.

Taking the h-covariant derivative of equation (14), with respect to x^m and x^l,respectively, we obtain the following result:

$$W^i_{jkh|m|l} = M^i_{jkh|m|l} + \tfrac{1}{12}\big[R\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big)\big]_{|m|l} + \tfrac{1}{3}\big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\,\big)_{|m|l}\ . \tag{39}$$

Using equations (5a) and (39) can be rewritten as follows:

$$W^i_{jkh|m|l} = M^i_{jkh|m|l} + \tfrac{1}{12}R_{|m|l}\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big) + \tfrac{1}{3}\big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\,\big)_{|m|l}\ . \tag{40}$$

Similarly, by applying equations (13) and (14) in (40), we obtain the result:

$$M^i_{jkh|m|l} + \tfrac{1}{12}R_{|m|l}\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big) + \tfrac{1}{3}\big(\delta^i_k\,R_{jh} - g_{jk}R^i_h\big)_{|m|l} = a_{ml}M^i_{jkh} + \tfrac{1}{12}a_{ml}R\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big) + \tfrac{1}{3}a_{ml}\big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\,\big) + b_{ml}\,\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) + \tfrac{1}{4}c_{ml}\big(W^i_k g_{jh} - W^i_h g_{jk}\big) + \tfrac{1}{4}\gamma_m\big(W^i_k g_{jh} - W^i_h g_{jk}\big)_{|l}\,.$$

alternatively, this can be expressed as:

$$M^i_{jkh|m|l} = a_{ml}M^i_{jkh} + b_{ml}\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) + \tfrac{1}{4}c_{ml}\big(W^i_k g_{jh} - W^i_h g_{jk}\,\big) + \tfrac{1}{4}\gamma_m\big(W^i_k g_{jh} - W^i_h g_{jk}\,\big)_{|l} - \tfrac{1}{12}R_{|m|l}\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big) - \tfrac{1}{3}\big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\,\big)_{|m|l} + \tfrac{1}{12}a_{ml}R\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big) + \tfrac{1}{3}a_{ml}\big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\,\big)\ . \tag{41}$$

This demonstrates that

$$M^i_{jkh|m|l} = a_{ml}M^i_{jkh} + b_{ml}\big(\delta^i_k g_{jh} - \delta^i_h g_{jk}\big) + \tfrac{1}{4}c_{ml}\big(W^i_k g_{jh} - W^i_h g_{jk}\big) + \tfrac{1}{4}\gamma_m\big(W^i_k g_{jh} - W^i_h g_{jk}\big)_{|l}\,. \tag{42}$$

If and only if

$$R_{|m|l}\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big) = a_{ml}R\big(\delta^i_k\,g_{jh} - \delta^i_h\,g_{jk}\big)\ \text{, and } \big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\,\big)_{|m|l} = a_{ml}\big(\delta^i_k\,R_{jh} - g_{jk}\,R^i_h\big)\,. \tag{43}$$

In conclusion the proof of theorem is completed, we can determine

Theorem 6. In the space $\mathrm{G^{2nd}M_{|h}\text{-}BRF_n}$, the concircular curvature tensor M^i_{jkh} defines a generalized birecurrent Finsler space if and only if the condition in equation (43) is satisfied.

By transvecting equation (41) with y^j and utilizing equations (10a), (4b), (1a) and (11a), we obtain the following result:

$$M^i_{kh|m|l} = a_{ml}M^i_{kh} + b_{ml}\big(\delta^i_k\,y_h - \delta^i_h\,y_k\big) + \tfrac{1}{4}c_{ml}\big(W^i_k y_h - W^i_h y_k\big) + \tfrac{1}{4}\gamma_m\big(W^i_k y_h - W^i_h y_k\big)_{|l} - \tfrac{1}{12}R_{|m|l}\big(\delta^i_k\,y_h - \delta^i_h\,y_k\big) - \tfrac{1}{3}\big(\delta^i_k\,H_h - y_k R^i_h\,\big)_{|m|l} + \tfrac{1}{12}a_{ml}R\,\big(\delta^i_k\,y_h - \delta^i_h\,y_k\big) + \tfrac{1}{3}a_{ml}\big(\delta^i_k\,H_h - \ y_k R^i_h\,\big)\,. \tag{44}$$

This demonstrates that

$$M^i_{kh|m|l} = a_{ml}M^i_{kh} + b_{ml}\,\big(\delta^i_k\,y_h - \delta^i_h\,y_k\big) + \tfrac{1}{4}c_{ml}\big(W^i_k y_h - W^i_h y_k\big) + \tfrac{1}{4}\gamma_m\big(W^i_k y_h - W^i_h y_k\big)_{|l}\ . \tag{45}$$

If and only if

$$R_{|m|l}\big(\delta^i_k\,y_h - \delta^i_h\,y_k\big) = a_{ml}R\big(\delta^i_k\,y_h - \delta^i_h\,y_k\big)\text{, and } \big(\delta^i_k\,H_h - y_k R^i_h\big)_{|m|l} = a_{ml}\big(\delta^i_k\,H_h - y_k R^i_h\big)\ . \tag{46}$$

Therefore, the proof of theorem is completed, we can say

Theorem 7. In the space $\mathrm{G^{2nd}M_{|h}\text{-}BRF_n}$, the h-covariant derivative of the second orders for the torsion tensor M^i_{kh} (concircular curvature tensor M^i_{jkh}) defines a generalized birecurrent Finsler space if and only if the condition in equation (46) is satisfied.

By transvecting condition (44) with y^k and applying along with equations (10b), (4a), (4b), (1b), (9a), (1c) and (11d), we obtain the following result:

$$M^i_{h|m|l} = a_{ml}M^i_h + b_{ml}\big(y^i\,y_h - \delta^i_h F^2\big) - \tfrac{1}{4}c_{ml}\big(W^i_h F^2\big) - \tfrac{1}{4}\,\gamma_m\big(W^i_h F^2\big)_{|l} - \tfrac{1}{12}R_{|m|l}\big(y^i\,y_h - \delta^i_h F^2\big) - \tfrac{1}{3}\big(y^i H_h - F^2 R^i_h\,\big)_{|m|l} - \tfrac{1}{12}a_{ml}R\big(y^i y_h - \delta^i_h F^2\big) + \tfrac{1}{3}a_{ml}\big(y^i H_h - F^2 R^i_h\big)\,. \tag{47}$$

This demonstrates that

$$M^i_{h|m|l} = a_{ml}M^i_h + b_{ml}\big(y^i y_h - \delta^i_h F^2\big) - \tfrac{1}{4}c_{ml}\big(W^i_h F^2\big) - \tfrac{1}{4}\,\gamma_m\big(W^i_h F^2\big)_{|l}\ . \tag{48}$$

If and only if

$$R_{|m|l}\big(y^i y_h - \delta^i_h F^2\big) = a_{ml}R\big(y^i y_h - \delta^i_h F^2\big)\text{, and } \big(y^i H_h - F^2 R^i_h\big)_{|m|l} = a_{ml}\big(y^i H_h - F^2\,R^i_h\big)\,. \tag{49}$$

Therefore, the proof of theorem is completed, we can say

Theorem 8. In the space $\mathrm{G^{2nd}M_{|h}\text{-}BRF_n}$, the projective deviation tensor M_h^i represents a generalized birecurrent Finsler space if the tensors $R(y^i y_h - \delta_h^i F^2)$ and $(y^i H_h - F^2 R_h^i)$ are generalized birecurrent Finsler space.

By contracting the indices i and h in the equations (41), (44) and (47), and utilizing equations $(n = 4)$, (2a), (1a), (1b), (9b), (9c), (9d), (11c), (11d) and (1c), along with the relations in equations (10c), (10d) and (10e), we obtain the following result:

$$M_{jk|m|l} = a_{ml}M_{jk} + (1-n)b_{ml}\, g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l} - \tfrac{1}{12}(1-n)\left(Rg_{jk}\right)_{|m|l} - \tfrac{1}{3}\left(R_{jk} - g_{jk}\,\mathrm{R}\right)_{|m|l} + \tfrac{1}{3}a_{ml}\left(R_{jk} - g_{jk}\,\mathrm{R}\right) + \tfrac{1}{12}(1-n)a_{ml}\left(\mathrm{R}g_{jk}\right). \tag{50}$$

This demonstrates that

$$M_{jk|m|l} = a_{ml}M_{jk} + (1-n)b_{ml}\, g_{jk} + \tfrac{1}{4}c_{ml}W_{jk} + \tfrac{1}{4}\gamma_m W_{jk|l}. \tag{51}$$

If and only if

$$\left(Rg_{jk}\right)_{|m|l} = a_{ml}\left(Rg_{jk}\right),$$
$$\left(R_{jk} - g_{jk}\,R\right)_{|m|l} = a_{ml}\left(R_{jk} - g_{jk}\,R\right). \tag{52}$$

And

$$M_{k|m|l} = a_{ml}M_k + (1-n)b_{ml}\, y_k - \tfrac{1}{3}(H_k - y_k R)_{|m|l} - \tfrac{1}{12}(1-n)(Ry_k)_{|m|l} + \tfrac{1}{3}a_{ml}(H_k - y_k R) + \tfrac{1}{12}(1-n)a_{ml}(Ry_k). \tag{53}$$

This demonstrates that

$$M_{k|m|l} = a_{ml}M_k + (1-n)b_{ml}\, y_k. \tag{54}$$

If and only if

$(Ry_k)_{|m|l} = a_{ml}(Ry_k)$, and
$$(H_k - y_k R)_{|m|l} = a_{ml}(H_k - y_k R). \tag{55}$$

In the last

$$M_{|m|l} = a_{ml}M + (1-n)b_{ml}\, F^2 - \tfrac{1}{3}(3H - F^2\mathrm{R})_{|m|l} - \tfrac{1}{12}(1-n)(RF^2)_{|m|l} + \tfrac{1}{3}a_{ml}(3H - F^2\mathrm{R}) + \tfrac{1}{12}(1-n)a_{ml}(RF^2). \tag{56}$$

This demonstrates that

$$M_{|m|l} = a_{ml}M + (1-n)b_{ml}\, F^2. \tag{57}$$

If and only if

$(RF^2)_{|m|l} = a_{ml}(RF^2)$, and
$$(3\mathrm{H} - F^2\mathrm{R})_{|m|l} = a_{ml}(3\mathrm{H} - F^2\mathrm{R}). \tag{58}$$

Therefore, the proof of theorem is completed, we can say

Theorem 9. In the space $\mathrm{G}^{2\mathrm{nd}}\mathrm{M}_{|\mathrm{h}}$-$\mathrm{BRF}_\mathrm{n}$, Ricci tensor M_{jk} , vector M_k and scalar M are defined in equations (51), (54) and (57), respectively, provided that the conditions in equations (52), (55) and (58) are satisfied.

By transvecting equation (41) with g_{ir} and utilizing equations (2b), (9d), and (10f), we obtain the following result:

$$M_{rjkh|m|l} = a_{ml}M_{rjkh} + b_{ml}\left(g_{rk}\, g_{jh} - g_{rh}g_{jk}\right) + \tfrac{1}{4}c_{ml}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|l} - \tfrac{1}{3}\left(g_{rk}R_{jh} - g_{jk}\,R_{rh}\right)_{|m|l} - \tfrac{1}{12}R_{|m|l}\left(g_{rk}\, g_{jh} - g_{rh}g_{jk}\right) + \tfrac{1}{3}a_{ml}\left(g_{rk}R_{jh} - g_{jk}\,R_{rh}\right) + \tfrac{1}{12}a_{ml}\mathrm{R}\left(g_{rk}\, g_{jh} - g_{rh}g_{jk}\right). \tag{59}$$

This demonstrates that

$$M_{rjkh|m|l} = a_{ml}M_{rjkh} + b_{ml}\left(g_{rk}\, g_{jh} - g_{rh}g_{jk}\right) + \tfrac{1}{4}c_{ml}\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right) + \tfrac{1}{4}\gamma_m\left(W_{rk}g_{jh} - W_{rh}g_{jk}\right)_{|l}. \tag{60}$$

If and only if

$R_{|m|l}\left(g_{rk}\, g_{jh} - g_{rh}g_{jk}\right) = a_{ml}\mathrm{R}\left(g_{rk}\, g_{jh} - g_{rh}g_{jk}\right)$, and
$$\left(g_{rk}R_{jh} - g_{jk}\,R_{rh}\right)_{|m|l} = a_{ml}\left(g_{rk}R_{jh} - g_{jk}\,R_{rh}\right). \tag{61}$$

Therefore, the proof of theorem is completed, we can say

Theorem 10. In the space $\mathrm{G}^{2\mathrm{nd}}\mathrm{M}_{|\mathrm{h}}$-$\mathrm{BRF}_\mathrm{n}$, associate tensor M_{jrkh} (Concircular curvature tensor M_{jkh}^i) characterizes a generalized birecurrent Finsler space, provided that the condition (61) is satisfied.

Numerical Illustration: CFD Analysis of Directional Flow Effects

To illustrate the potential applicability of generalized second-order recurrent geometric structures, a numerical investigation based on computational fluid dynamics (CFD) is presented. Although the theoretical framework developed in this paper is purely geometric, CFD provides a practical setting in which anisotropic and direction-dependent behavior can be observed and interpreted.

The numerical study considers the incompressible flow of crude oil through pipe elbows with different curvature radii. Examined are four elbow configurations with bend radii of 1.5D, 2D, 2.5D, and 3D, where D is the pipe diameter. Typical operating circumstances are represented by imposing inlet velocities between 0.5 and 2 m/s. A turbulence model appropriate for near-wall flow behavior is used to solve the governing equations of momentum and mass conservation. To guarantee solution stability and convergence, mesh refinement is carried out using an organized computational mesh.

The velocity field clearly depends on elbow geometry, according to the numerical data. Strong asymmetry is seen in the flow at lower bend radii, with higher velocity magnitudes concentrated around the elbow's outer curvature. As the input velocity rises, this directional bias becomes more pronounced. Elbows with greater curvature radii, on the other hand, show less flow separation and smoother velocity patterns.

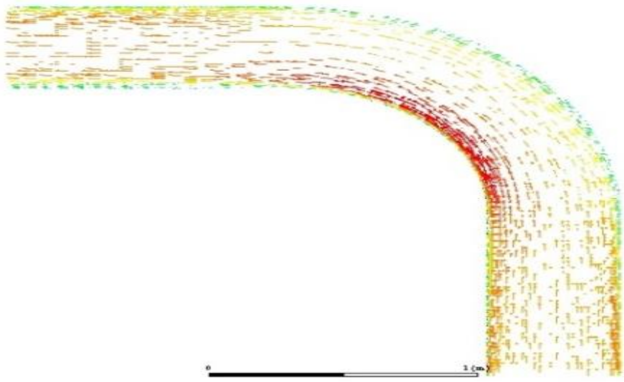
Similar trends are seen in wall shear stress, which rises dramatically in areas with the largest curvature and flow acceleration effects. Smoother bends lessen the impact of higher shear concentrations caused by sharper elbows. These findings demonstrate how flow behavior is sensitive to directional limitations and geometric curvature.

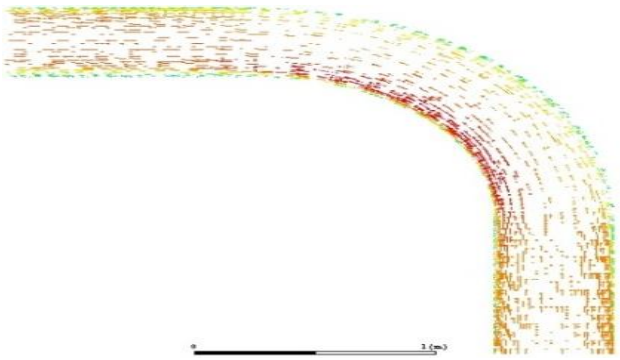
Table 1: Maximum shear stress

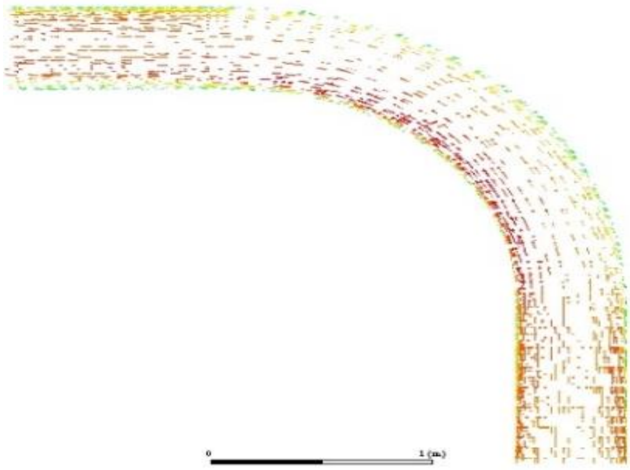
Velocity, V (m/s)	Maximum shear stress, t (Pa)			
	1.5 D	2 D	2.5 D	3 D
0.5	0.008435	0.008435181	0.008435171	0.008435165
1	0.02331102	0.2331098	0.02331096	0.02331094
1.5	0.04228811	0.04228804	0.042288	0.04228797
2	0.06451498	0.06451481	0.06451469	0.06451461

Geometric Interpretation

Geometrically speaking, the observed shear concentration and flow asymmetry represent anisotropic behavior similar to that explained by generalized recurrent structures in Finsler geometry. Higher-order geometric constraints may affect transport phenomena, as demonstrated physically by the dependence of flow characteristics on direction and curvature. While no direct equivalence is claimed, the numerical results support the relevance of generalized second-order recurrence as a conceptual framework for modeling systems with intrinsic directional dependence.







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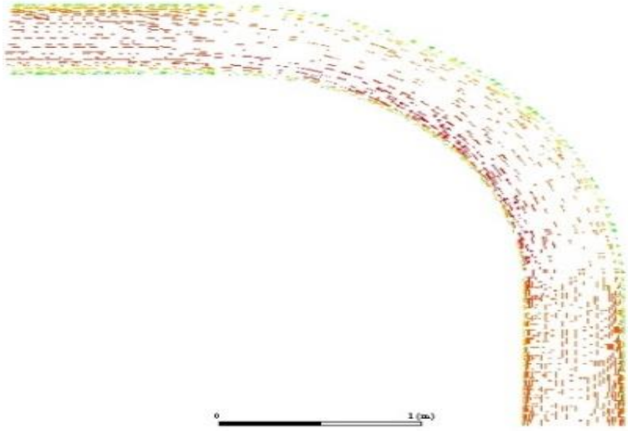
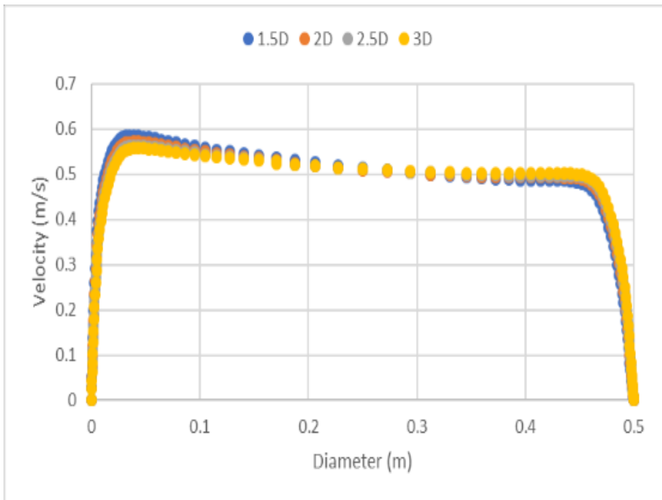
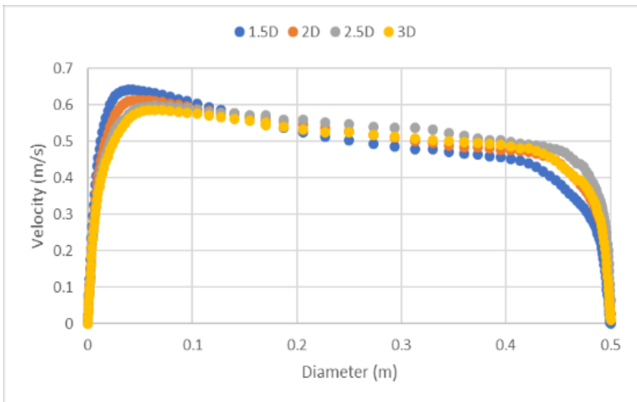


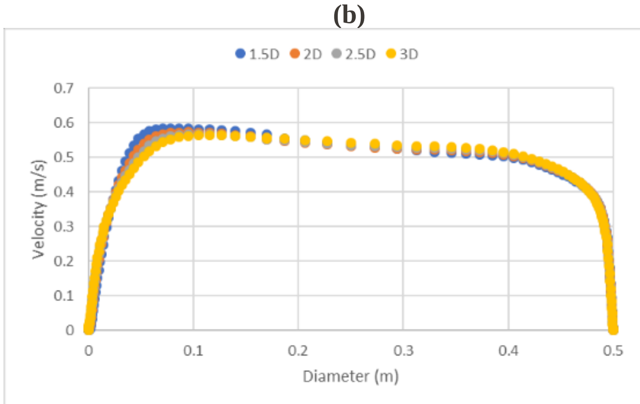
Figure 1: Flow's effect on elbow design (a) 1.5 D, (b) 2 D, (c) 2.5 D, and (d) 3 D



(a)



(b)



(c)

Figure 2: shows the velocity curve vs diameter at v = 0.5 m/s at

angles of 0°, 45°, and 90°

Results and Discussion

The framework for generalized second-order recurrent and birecurrent conditions in Finsler geometry is first established.

In particular, we define the classes of generalized second-order concircular-birecurrent $G^{2nd}M_{|h}$ - BRF_n and generalized second-order Weyl-recurrent $G^{2nd}M_{|h}$ - RF_n spaces. The structure of Weyl and concircular curvature is influenced by recurrence conditions, as shown by the derivation of several significant relations between curvature tensors using rigorous tensorial calculations.

Important outcomes consist of the following:

1. Characterization of generalized recurrence: Weyl and concircular curvature tensors are used to determine the necessary and sufficient conditions for a Finsler space to admit generalized second-order recurrence.
2. Interdependence of curvature tensors: It is demonstrated that the Weyl curvature tensor and the concircular curvature tensor are closely related under certain recurrence circumstances, resulting in new structural constraints on the geometry.
3. Extension of classical results: The results demonstrate the wider applicability of second-order conditions by generalizing previous conclusions on first-order recurrent and birecurrent Finsler spaces.
4. Possible cosmological ramifications: Despite the analysis's theoretical nature, the geometric structures found could be used as mathematical models to explain anisotropic spacetime geometries in cosmology.

These contributions open up new avenues for the use of higher-order recurrence to mathematical physics and improve our knowledge of recurrent structures in Finsler geometry.

Conclusion

With a focus on their connections to Weyl's projective and concircular curvature tensors, this paper provides a thorough theoretical explanation of generalized second-order recurrent and birecurrent Finsler spaces. The necessary and sufficient conditions for the creation of new classes of generalized Weyl-recurrent and concircular-birecurrent spaces were established. The structural and geometric framework of higher-order recurrence in Finsler geometry is enhanced by these conclusions, which go beyond conventional findings. A CFD-based numerical study of directed flow in curved pipe elbows was carried out to demonstrate their practical significance. The simulations showed that curvature and directional limitations have a significant impact on velocity distribution, flow asymmetry, and wall shear stress. Similar to generalized recurring structures, this behavior offers a significant physical interpretation of anisotropy. As a result, the study emphasizes how these geometric models may be used in fluid mechanics, engineering systems, and cosmological theories that deal with anisotropic and curvature-dependent phenomena.

Author Contributions: Al-Qashbari: Conceptualization, methodology and supervision; **Muthanna** and **Hadi**: Validation, formal analysis and investigation; **Al-Qashbari**, **AL-ssallal** and **Arroussi**: Writing original draft; **Chekalil** and **Mandal**: Review and editing and visualization. All authors reviewed and approved the final version of the manuscript.

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