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Analytical Solution and Numerical Simulation of the Two-Dimensional Time-Fractional Reaction-Diffusion Brusselator System

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ABSTRACT

In this study, the Natural decomposition method is introduced to solve the two-dimensional nonlinear time-fractional reaction-diffusion Brusselator system. The fractional derivatives are described in the Caputo sense. The closed-form solution, as it is a vital momentary in solving sophisticated nonlinear differential equations, was obtained and to be investigated thoroughly in the present study. Additionally, the approximate semi-analytic solution is presented as a rapidly convergent series, which is proved via tables and illustrations. The nonlinearity of the equations is decomposed using the Adomian polynomials. All computations were executed via Mathematica symbolic computational package.

الحل التحليلي والمحاكاة العددية لنظام بروسلاتور التفاعلي-الانتشاري الزمني الكسري ثنائي البعد

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الكلمات المفتاحية

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الملخص

في هذه الدراسة، تم تقديم طريقة التحلل الطبيعي لحل نظام بروسلاتور الزمني الكسري للتفاعل والانتشار. حيث وُصفت المشتقات الكسرية بمعنى كابوتو. تم الحصول على الحل المغلق والذي يعتبر لحظة حيوية في حل المعادلات التفاضلية الغير خطية المعقدة. بالإضافة لذلك، عُرض الحل شبه التحليلي التقريبي كمتسلسلة سريعة التقارب وتم اثباتها عبر الجداول والرسومات. تم معالجة اللاخطية في المعادلات باستخدام حدوديات ادوميان، وتم تنفيذ جميع الحسابات باستخدام الحوسبة الرمزية ماثيماتيكا.

Introduction

The reaction-diffusion equations are well-known for describing various phenomena arises in engineering, biology, ecology, physics, chemistry, … etc [1,2]. Recently, these equations are preferably modelled using fractional partial differential equations (FPDEs) for the sake of the information provided by the heredity properties and memory effects [3–5]. The present study focuses on one of the most eminent equations that presents the reaction-diffusion process in biological and chemical phenomena, the fractional Brusselator System, which models the enzymatic reactions and triple collisions of atomic oxygen forming the ozone, as an instance [6,7].

In the last few years, a considerable amount of attraction was given to obtain the analytical, semi-analytical and approximate solution of reaction-diffusion equations either with fractional order or the integer one by using trustworthy methods, namely: the Reduced differential transform method (RDT) [7,8], the expansion approach; the $\exp(-\Omega(\eta))$-expansion approach [9], the Finite difference methods [10–13], The finite element method [14] , differential quadrature method (DQM) [15], the Homotopy analysis method (HAM) [16], the Optimal Homotopy asymptotic method (OHAM) [17], the Homotopy Perturbation method (HPM) [18,19], the New Iterative method alongside the Residual power series method (RPSM) [20–22], and Adomian decomposition method (ADM) [21–34].

Further, solving FPDEs by using Integral transforms-based schemes, such as the Natural transform (NT) coupled with Adomian decomposition method (ADM) [30,32,33], the Laplace transform (LT) coupled with Homotopy Perturbation method (HPM) and ADM [26,27,29,34], the Yang transform (YT) coupled with the Residual Power series method (RPSM) [25], and the Elzaki transform (ET) coupled with RPSM [28], have been proven to be robust and computationally efficient. The method of Natural decomposition method (NDM) was firstly conceived and developed later in literature, a decade ago by Rawashdeh and

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Maitama [35] as a combination of NT and Adomian polynomials to obtain the exact solution of coupled systems of nonlinear partial differential equations of integer derivative order (CSLNPDEs). Additionally, their study has proved significant advancements compared to the other semi-analytical ADM, LDM, HAM and VIM. In a recent study done by Ahmed et al. [31] the NDM was implemented on the two-dimensional fisher equation of Caputo's fractional order, as this equation is widely applied in autocatalytic chemical reactions and chemical kinetics. The method was proved to be valid and proficient in solving two-dimensional FPDEs. Proceeding further, in the present article, we aim to extend the work of Ahmed et al. [31] to investigate the analytical solution of a two-dimensional fractional order system by using the method of Natural Decomposition and comparing the results with those reported by the method of RDT [7]. This paper addresses the two-dimensional time-fractional Brusselator system as [7]:

$$\begin{aligned}\mathcal{D}_t^{\gamma} v(x,y,t) &= \lambda\left(\mathcal{D}_x^2 v(x,y,t)+\mathcal{D}_y^2 v(x,y,t)\right)\\ &-(a+1)\,v(x,y,t)+v^2(x,y,t)w(x,y,t)+b\\ \mathcal{D}_t^{\beta} w(x,y,t) &= \lambda\left(\mathcal{D}_x^2 w(x,y,t)+\mathcal{D}_y^2 w(x,y,t)\right)\\ &-v^2(x,y,t)w(x,y,t)+a\,v(x,y,t)\end{aligned}\tag{1}$$

where $0<\gamma,\beta\le 1$ and $t>0$, with the initial conditions:

$$\begin{aligned}v(x,y,0)&=\varphi\,(x,y)\\ w(x,y,0)&=\psi(x,y)\end{aligned}\tag{2}$$

In which $\mathcal{D}_t^{\gamma}$ and $\mathcal{D}_t^{\beta}$ denote the Caputo's fractional derivatives operators of orders $0\le\alpha,\beta\le 1$, the constant λ is the coefficient of diffusion, a,b are the inlet reactant concertation constants. Both functions $v(x,y,t)$ and $w(x,y,t)$ represent the dimensionless concentration of the two reactants.

This article is organized to offer basic concepts as well as preliminaries of Fractional Calculus (FC) and the Natural Integral Transform in the first section. Besides, the methodology of the fractional NDM is presented in great details afterwards. Then we have presented a meticulous simulation and interpretation of the effect of the fractional order. Finally, the achievements of this scientific article were provided in conclusion section.

Preliminaries

In this section we recall the most essential definitions and properties of FC and the Natural transform [3,4,32,36,37].

Definition 1. [3,4]

Let g be an analytic function of t in $L([0,1],\mathbb{R})$, the integral operator in Rieman-Liouville (R-L) approach of order γ, is:

$$\mathcal{J}_o^{\gamma} g(t)=\begin{cases} g(t) & ,\gamma=0\\ \dfrac{1}{\Gamma(\gamma)}\displaystyle\int_0^t (t-\zeta)^{\gamma-1} g(\zeta)d\zeta & ,\gamma\ne 0\end{cases}\tag{3}$$

where the Gamma function is defined as the improper integral $\Gamma(\varepsilon)=\int_0^{\infty} e^{-\tau}\tau^{\varepsilon-1}\,d\tau$, $\varepsilon\in\mathbb{C}$. It is worth mentioning that the Integral $\int_0^t (t-\zeta)^{\gamma-1} g(\zeta)d\zeta$ in Eq. (3) is presumed to be convergent.

Definition 2. [3,4]

The Caputo's fractional derivative of order γ of a function g of t, that is continuous on $\mathbb{C}_{-1}^{n}$, such that $n\in\mathbb{N}\cup\{0\}$ as t tends to infinity, is given as:

$$\mathcal{D}_t^{\gamma} g(t)=\begin{cases}\mathcal{J}^{n-\gamma} g^{(n)}(t) & ,n-1<\gamma\le n\\ d^n g(t)/d\tau^n & ,\gamma=n\end{cases}\tag{4}$$

where $\mathcal{J}^{n-\gamma} g^{(n)}$ is the R-L Integral of $g^{(n)}(t)$ defined in Eq (3).

Definition 3. [37]

The two parameter Mittag-Leffler function, is defined as:

$$E_{\gamma,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\,\gamma+\beta)}\tag{5}$$

where the fractional orders are $0<\gamma,\beta<1$ and $t\in\mathbb{C}$. By setting $\beta=1$, the $E_{\gamma,\beta}(t)$ in Eq. (5) yields the one parameter Mittag-Leffler function of the formula $E_{\gamma}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\,\gamma+1)}$. Also, by setting $\gamma=\beta=1$, the Mittag-Leffler function $E_{\gamma,\beta}(t)$ in Eq. (5) reduced to the classical exponential function $E_{1,1}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{t^k}{k!}=e^t$.

Next, we provide the basics of the Natural integral transform throughout definitions and lemmas.

Definition 4. The NT (see Belgacem *et al.* [32]) of a function $g(x_i,t)$ is defined as:

$$\begin{aligned}&\mathcal{N}^{+}[g(x_i,t)]=\mathcal{G}(x_i,s,u)=\\ &\int_0^{\infty} e^{-st} g(x_i,ut)dt=\frac{1}{u}\mathcal{L}\{g(x_i,t)\}\ ,s,u\in\mathbb{R}^{+}\end{aligned}\tag{6}$$

where $\mathcal{L}$ is the Laplace transform, s and u are the NT parameters. If u equals 1 the Laplace transform is obtained and the Sumudu transform is obtained for s equals 1.

Definition 5. The inverse NT (see Belgacem *et al.* [32]) of Eq. (6) is given by:

$$\begin{aligned}&\mathcal{N}^{-}[\mathcal{G}(x_i,s,u)]=g(x_i,t)\\ &=\frac{1}{2\pi i}\int_{q-i\infty}^{q+i\,\infty} e^{-st}\mathcal{G}(x_i,s,u)\ ds\end{aligned}\tag{7}$$

where $q\in\mathbb{R}$ such that $s=x+iy$ in the complex plane.

Lemma 1. The NT [32] of the R-L fractional derivative of a function $g(x_i,t)$ with respect to the variable t has the form:

$$\begin{aligned}&\mathcal{N}^{+}\left[\mathcal{D}_t^{\gamma} g(x_i,t)\right]=\frac{s^{\gamma}}{u^{\gamma}}\mathcal{G}(x_i,s,u)\\ &-\sum_{k=0}^{n-1}\frac{s^k}{u^{\gamma-k}}\left[\lim_{t\to 0}\mathcal{D}_t^{\gamma-(k+1)} g(x_i,t)\right]\end{aligned}\tag{8}$$

Lemma 2. The NT of the Caputo fractional derivative, [32], of a function $g(x_i,t)$ with respect to the variable t is defined as:

$$\begin{aligned}&\left[\mathcal{D}_t^{\gamma} g(x_i,t)\right]=\frac{s^{\gamma}}{u^{\gamma}}\mathcal{G}(x_i,s,u)\\ &-\sum_{k=0}^{n-1}\frac{s^{k-(k+1)}}{u^{\gamma-k}}\left[\lim_{t\to 0}\mathcal{D}_t^{\gamma} g(x_i,t)\right]\end{aligned}\tag{9}$$

Lemma 3. The NT of fractional derivative, [32], of a function $g(x_i,t)$ with respect to the variable x is defined as:

$$\mathcal{N}^{+}\left[\mathcal{D}_x^{\gamma} g(x_i,t)\right]=\mathcal{D}_x^{\gamma}\mathcal{G}(x_i,s,u)\tag{10}$$

The Fractional Natural Decomposition Method (NDM)

In this section, we consider the following multi-dimensional fractional partial differential system of n equations:

$$\mathcal{D}_t^{\gamma_j} v_i(\mathbf{X},t)+Rv_i(\mathbf{X},t)+Nv_i(\mathbf{X},t)=g_i(\mathbf{X},t)\tag{11}$$

where the fractional $m-1\le\gamma\le m$, $m\in\mathbb{N}$, $i=1,2,\dots,n$.

Subjected to the initial conditions (ICs)of the form:

$$\mathcal{D}_t^{\gamma} v_i(\mathbf{X},t)=f_{ik_i}(\mathbf{X})\tag{12}$$

in which $k_i=0,1,\dots,m_i-1$, $m_i\in\mathbb{N}$, $i,j=1,2,\dots,n$.

Where $\mathcal{D}_t^{\gamma}=\frac{\partial^{\gamma}}{\partial t^{\gamma}}$ is the Caputo fractional derivatives of order

$\gamma_j \quad (m-1 \le \gamma_j \le m), \quad \mathbf{X} = (x, y, z)^T. \quad Rv_i(\mathbf{X}, t)$ and $Nv_i(\mathbf{X}, t)$ are the linear and the nonlinear operators. As, $g_i(\mathbf{X}, t)$ is a continuous function represents the source term. The methodology of NDM consists of the following four steps:

Step 1.

Applying the NT on both sides of Eq. (11), and utilizing its properties, we get:

$$\mathcal{N}^+\left[\mathcal{D}_t^{\gamma_j} v_i(\mathbf{X}, t)\right] = \frac{f_{ik_i}(\mathbf{X})}{s} + \frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+[g_i(\mathbf{X}, t)]x - \frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+[Rv_i(\mathbf{X}, t) + Nv_i(\mathbf{X}, t)] \tag{13}$$

Step 2.

Operating with the inverse NT on both sides of Eq. (13), yields

$$v_i(\mathbf{X}, t) = f_{ik_i}(\mathbf{X}) + \mathcal{N}^-\left[\frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+[g_i(\mathbf{X}, t)]\right]x - \mathcal{N}^{-1}\left[\frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+[Rv_i(\mathbf{X}, t) + Nv_i(\mathbf{X}, t)]\right] \tag{14}$$

Step 3.

Decomposing the solution and the nonlinear term according to Adomian [38,39], one can formulate:

The infinite series form of the solution in Eq. (14), to have:

$$v_i(\mathbf{X}, t) = \sum_{n=0}^{\infty} v_{in}(\mathbf{X}, t) \tag{15}$$

The decomposition of the nonlinear term $v_i(\mathbf{X}, t)$ is:

$$Nv_i(\mathbf{X}, t) = \sum_{n=0}^{\infty} \mathcal{A}_{in} \tag{16}$$

where $\mathcal{A}_{in}$ is the Adomian polynomials [39]. In view of the above Eq. (15) and Eq. (16), the solution of $v_i(\mathbf{X}, t)$ in Eq. (11) can be written as:

$$\sum_{n=0}^{\infty} v_{in}(\mathbf{X}, t) = f_{ik_i}(\mathbf{X}) + \mathcal{N}^-\left[\frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+[g_i(\mathbf{X}, t)]\right] - \mathcal{N}^{-1}\left[\frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+\left[R\sum_{n=0}^{\infty} v_{in}(\mathbf{X}, t) + N\sum_{n=0}^{\infty} \mathcal{A}_{in}\right]\right] \tag{17}$$

Step 4.

Constructing the recurrence relation and the approximate solution, to get:

$$\begin{cases} v_{i0}(\mathbf{X}, t) = f_{ik_i}(\mathbf{X}) + \mathcal{N}^-\left[\frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+[g_i(\mathbf{X}, t)]\right] \\ v_{ij+1}(\mathbf{X}, t) = -\mathcal{N}^{-1}\left[\frac{u^{\gamma_j}}{s^{\gamma_j}} \mathcal{N}^+\left[Rv_{ij}(\mathbf{X}, t) + N\mathcal{A}_{ij}\right]\right] \end{cases} \tag{18}$$

where $j = 0,1,2, \ldots$. The m^th^ -approximate solution of the system (8), as the finite series of equation (12), have the form:

$$v_i(\mathbf{X}, t) = v_{i0}(\mathbf{X}, t) + v_{i1}(\mathbf{X}, t) + v_{i2}(\mathbf{X}, t) + \cdots + v_{im}(\mathbf{X}, t) = \sum_{n=0}^{m} v_{in}(\mathbf{X}, t) \tag{19}$$

Application, Numerical Simulation, and Interpretation

In this section, we run a numerical simulation on the two-dimensional time-fractional Brusslator systems [7], and investigate the memory effect of the Caputo's fractional derivative over time.

Application

Consider the initial value problem (IVP) of the Two-dimensional Brusselator system Eq. (1) and Eq. (2) and for the purpose of the numerical simulation and comparison, the parameters of the Brusselator systems [7] are taken as $a = 1, b = 0, \lambda = 0.25$ and $\varphi(x, y) = e^{-x-y}$, $\psi(x, y) = e^{x+y}$.

The corresponding Natural equation of Eq. (1) and by considering the ICs (2) is:

$$\begin{cases} v(x, y, t) = v(x, y, 0) + b \\ +\mathcal{N}^-\left[\frac{u^\gamma}{s^\gamma} \mathcal{N}^+\left[\lambda\left(\mathcal{D}_x^2 v + \mathcal{D}_y^2 v\right) - (a+1)v + v^2 w\right]\right] \\ w(x, y, t) = w(x, y, 0) \\ +\mathcal{N}^-\left[\frac{u^\beta}{s^\beta} \mathcal{N}^+\left[\lambda\left(\mathcal{D}_x^2 w + \mathcal{D}_y^2 w\right) - v^2 w + av\right]\right] \end{cases} \tag{20}$$

Following the decomposition procedure in step **3** of section **3**, along with step **4**, we obtain the recurrence relation of the Natural Eq. (20) as:

$$\begin{cases} v_0(x, y, t) = v(x, y, 0) + b = \varphi(x, y) \\ w_0(x, y, t) = w(x, y, 0) = \psi(x, y) \end{cases} \tag{21a}$$

$$\begin{cases} v_{j+1}(x, y, t) = \\ \mathcal{N}^-\left[\frac{u^\gamma}{s^\gamma} \mathcal{N}^+\left[\lambda\left(\mathcal{D}_x^2 v_j + \mathcal{D}_y^2 v_j\right) - (a+1)v_j + A_j\right]\right] \\ w_{j+1}(x, y, t) = \\ \mathcal{N}^-\left[\frac{u^\beta}{s^\beta} \mathcal{N}^+\left[\lambda\left(\mathcal{D}_x^2 w_j + \mathcal{D}_y^2 w_j\right) - A_j + av_j\right]\right] \end{cases} \tag{21b}$$

where $\mathcal{A}_j = \frac{1}{j!}\frac{d^j}{d\lambda^j}\left[N\sum_{i=1}^{j} \lambda^i v^i(x, y, t)\right]_{\lambda=0}$, and the subsequent few terms are:

$$\begin{cases} v_0(x, y, t) = \varphi(x, y) = e^{-x-y} \\ w_0(x, y, t) = \psi(x, y) = e^{x+y} \end{cases}$$

$$\begin{cases} v_1(x, y, t) = -\frac{0.5\, t^\gamma}{\Gamma(\gamma+1)} e^{-x-y} \\ w_1(x, y, t) = \frac{0.5\, t^\beta}{\Gamma(\beta+1)} e^{x+y} \end{cases}$$

$$\begin{cases} v_2(x, y, t) = -\frac{0.25\, t^{2\gamma}}{\Gamma(2\gamma+1)} e^{-x-y} \\ w_2(x, y, t) = \frac{0.25\, t^{2\beta}}{\Gamma(2\beta+1)} e^{x+y} \end{cases}$$

$$\vdots$$

$$.etc.$$

Therefore, the m^th^ approximate solution by the virtue of equation (17) is:

$$\begin{cases} v(x, y, t) = \sum_{j=0}^{m} v_j(x, y, t) = v_0(x, y, t) \\ +v_1(x, y, t) + \cdots + v_m(x, y, t) \\ w(x, y, t) = \sum_{j=0}^{m} w_j(x, y, t) = w_0(x, y, t) \\ +w_1(x, y, t) + \cdots + w_m(x, y, t) \end{cases} \tag{22}$$

As the value of m tends to infinity and by considering Definition **3** of the Mittag-Leffler function, the approximate solution of the time-fractional Brusselator system Eq. (1) have the form:

$$\begin{cases} v(x,y,t) = e^{-x-y}\left(1+\sum_{k=1}^{\infty}\frac{(-(t/2)^{\gamma})^{k}}{\Gamma(k\gamma+1)}\right) \\ \qquad = e^{-x-y}E_{\gamma}(-(t/2)^{\gamma}) \\ w(x,y,t) = e^{x+y}\left(1+\sum_{k=1}^{\infty}\frac{\left(-(t/2)^{\beta}\right)^{k}}{\Gamma(k\beta+1)}\right) \\ \qquad = e^{x+y}E_{\beta}\left((t/2)^{\beta}\right) \end{cases} \tag{23}$$

Noting that, if $\gamma = \beta = 1$, this yields:

$$\begin{cases} v(x,y,t) = e^{-x-y}\left(1-t+\frac{(t/2)^{2}}{2!}-\frac{(t/2)^{3}}{3!}+\cdots\right) \\ \qquad = e^{-x-y-t/2} \\ w(x,y,t) = e^{x+y}\left(1+t+\frac{(t/2)^{2}}{2!}+\frac{(t/2)^{3}}{3!}+\cdots\right) \\ \qquad = e^{x+y+t/2} \end{cases} \tag{24}$$

which gives the exact solution of the Brusselator system Eq. (1) as given in [19].

Numerical Simulation, and Interpretation

In this subsection, as we mentioned earlier the systems' constants represented in the Bursselator system Eq. (19) were assigned the numerical values $a = 1, b = 0, \lambda = 0.25$ and the functions of ICs are $\varphi\,(x,y) = e^{-x-y}$, $\psi(x,y) = e^{x+y}$.

In Fig. 1, the 3D profiles of the 5th-term approximate $v_5(x,y,t)$ and $w_5(x,y,t)$ in comparison with the actual solution Eq. (24) are presented, as the fractional order derivatives were appointed to the values $\gamma = 0.7, 0.8, 0.9, 1$. The illustrations in Fig. 1 demonstrate the rapid convergence of the approximate solution to the actual one as γ tends to the integer order $\gamma = 1$. Further still, Tables 1 and 2 detail out a numerical representation of the comparison of $v_5(x,y,t)$ and $w_5(x,y,t)$ over time $t = 0.1, 0.5, 1.5, 2$.

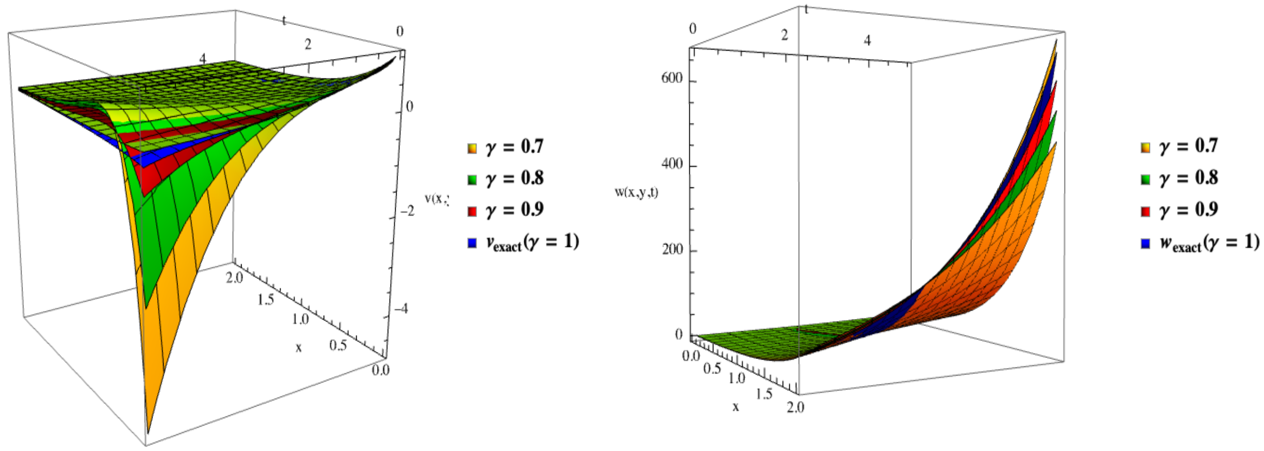

Figure 1: The 3D plot of analytical solutions $\boldsymbol{v(x,y,t)}$ and $\boldsymbol{w(x,y,t)}$ (25) when $\boldsymbol{\gamma = \beta = 1}$ vs. the 5th approximates $\boldsymbol{v_5(x,y,t)}$ and $\boldsymbol{w_5(x,y,t)}$ when $\boldsymbol{\gamma = \beta = 0.7, 0.8, 0.9}$,

Table 1: Numerical results of the Brussolatr system of the function $(\boldsymbol{x,y,t})$, at $\boldsymbol{x = y = 1.5, t = 0.1, 0.5, 1.5, 2}$

	$v_5\,(x,y,t)$				$v(x,y,t)$
t	$\boldsymbol{\gamma = 0.7}$	$\boldsymbol{\gamma = 0.8}$	$\boldsymbol{\gamma = 0.9}$	$\boldsymbol{\gamma = 1}$	$(\boldsymbol{\gamma = 1})$
.1	0.0447196	0.0457673	0.0466444	0.0473589	0.0473589
.5	0.0367394	0.0372014	0.0379134	0.0387742	0.0387742
1.5	0.0320898	0.0284164	0.0255924	0.0235066	0.0235177
2	0.0344962	0.0280868	0.0226622	0.0182553	0.0183156

Table 2: Numerical results of the Brussolatr system of the function $(\boldsymbol{x,y,t})$, at $\boldsymbol{x = y = 1.5, t = 0.1, 0.5, 1.5, 2}$

$w_5\,(x,y,t)$				$w(x,y,t)$
$\boldsymbol{\beta = 0.7}$	$\boldsymbol{\beta = 0.8}$	$\boldsymbol{\beta = 0.9}$	$\boldsymbol{\beta = 1}$	**(β = 1)**
22.4612	21.8861	21.4488	21.1153	21.1153
28.7298	27.6157	26.6422	25.7903	25.7903
45.6188	44.6673	43.6156	42.5155	42.5211
56.0582	55.8458	55.3168	54.5657	54.5982

In Tables 1 and 2, by observing the provided values of the numerical and the actual solution, one can confirm that they are in a strong agreement and converges to each other as t gets smaller. Incorporating this numerical investigation further by intensifying comparative 3D profile illustrations in Figs. 2 and 3, where the exact $v(x,y,t)/w(x,y,t)$, the approximate $v_5\,(x,y,t)/\ w_5(x,y,t)$ and the absolute error were imposed in one graph for each fractional order γ, *respectively*. As a result, the memory effect of the fractional order of the Brusselator system Eq. (1) were clearly shown to which γ gradually tends the integer order $\gamma = 1$ and the NDM has shown a great potential in approximating the actual solution in such a low order of the truncated-series.

Moreover, Numerical comparison of $v(x,y,t)/\ w(x,y,t)$ Eq. (24) and $v_5\,(x,y,t)/\ w_5\,(x,y,t)$ when $\gamma = 0.7, 0.8, 0.9, 1$. are exhibited in Tables 3 and 4. The absolute errors validate that the performance of the NDM is decent for what we have presented in Figs 2 and 3

Advancing this numerical investigation, with the NDM the tenth approximate solution results are compared to the numerical results of the RDT method [7], in Tables 5 and 6. Besides, Fig. 4 illustrates the absolute error of the 10th- term series solution of the Brusselator system Eq. (1).

The error analysis, in both, demonstrates that the NDM is a reliable technique for investigating the actual solution and the approximate one in such multi-dimensional models.

Table 3: Numerical comparison of $\boldsymbol{v(x,y,t)}$ (24) and $\boldsymbol{v_5\,(x,y,t)}$ when $\boldsymbol{\gamma} = 0.7, 0.8, 0.9, 1$.

		$\lvert v - v_5\rvert$			
	$\boldsymbol{t}$	$\boldsymbol{\gamma = 0.7}$	$\boldsymbol{\gamma = 0.8}$	$\boldsymbol{\gamma = 0.9}$	$\boldsymbol{\gamma = 1}$
$\boldsymbol{x = y = 1}$	0.5	4.1945×10^{-3}	3.15896×10^{-3}	1.67325×10^{-3}	4.4301×10^{-8}
	1	6.74841×10^{-3}	3.36507×10^{-3}	1.23626×10^{-3}	2.7396×10^{-6}
	1.5	2.09151×10^{-2}	1.22001×10^{-2}	5.32804×10^{-3}	3.01795×10^{-5}
	2	3.86094×10^{-2}	2.34968×10^{-2}	1.05576×10^{-2}	1.64131×10^{-4}
$\boldsymbol{x = y = 1.5}$	0.5	1.53811×10^{-3}	1.16077×10^{-3}	6.15285×10^{-4}	1.62974×10^{-8}
	1	2.42527×10^{-3}	1.43202×10^{-3}	6.27124×10^{-4}	1.07278×10^{-12}
	1.5	7.85114×10^{-3}	4.55115×10^{-3}	1.97788×10^{-3}	1.11024×10^{-5}
	2	1.46007×10^{-2}	8.82174×10^{-3}	3.939×10^{-3}	6.03805×10^{-5}
$\boldsymbol{x = y = 2}$	0.5	5.6559×10^{-4}	4.26956×10^{-3}	2.26337×10^{-4}	5.99548×10^{-9}
	1	9.31306×10^{-4}	4.61646×10^{-3}	1.68862×10^{-4}	3.70764×10^{-7}
	1.5	2.89612×10^{-3}	1.67742×10^{-3}	7.28512×10^{-4}	4.08435×10^{-6}
	2	5.39116×10^{-3}	3.25423×10^{-3}	1.45183×10^{-3}	2.22127×10^{-5}

Table 4: Numerical comparison of $w(x,y,t)$ (24) and $w_5(x,y,t)$ when $\beta = \gamma = 0.7, 0.8, 0.9, 1$.

		$\lvert w - w_5\rvert$			
	$\boldsymbol{t}$	$\boldsymbol{\beta = 0.7}$	$\boldsymbol{\beta = 0.8}$	$\boldsymbol{\beta = 0.9}$	$\boldsymbol{\beta = 1}$
$\boldsymbol{x = y = 1}$	0.5	7.14079×10^{-1}	4.20305×10^{-1}	1.85709×10^{-1}	2.59787×10^{-6}
	1	1.45978×10^{-2}	6.39062×10^{-3}	2.20966×10^{-3}	1.72564×10^{-4}
	1.5	2.19558	1.26531	5.51691×10^{-1}	2.04222×10^{-3}
	2	6.79575	3.826	1.64291	1.19345×10^{-2}
$\boldsymbol{x = y = 1.5}$	0.5	1.94605	1.14447	5.05384×10^{-1}	7.06174×10^{-6}
	1	1.53185×10^{-2}	5.9807×10^{-3}	2.06049×10^{-3}	4.69078×10^{-4}
	1.5	5.90394	3.40632	1.48706	5.55132×10^{-3}
	2	1.83416×10^{1}	1.03274×101	4.43647	3.24414×10^{-2}
$\boldsymbol{x = y = 2}$	0.5	5.29177	3.11172	1.37399	1.91958×10^{-5}
	1	3.254×10^{-2}	1.20192×10-^{2}	4.13755×10^{-3}	1.27509×10^{-3}
	1.5	1.60244×10^{1}	9.24693	4.03757	1.509×10^{-2}
	2	4.9808×10^{1}	2.80454×10^{1}	1.20486×10^{1}	8.81848×10^{-2}

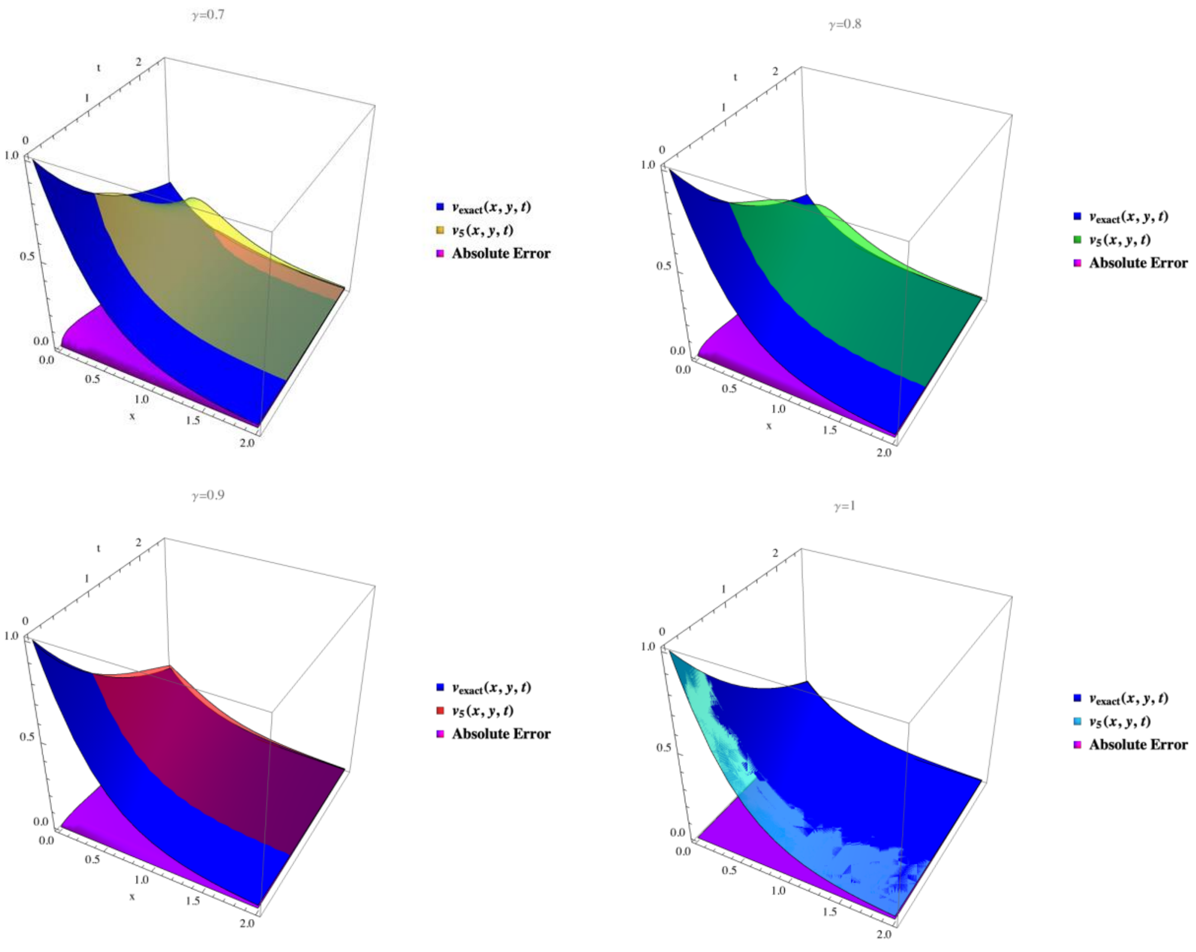

Figure 2: The 3D plot comparison surfaces of exact solutions (24) and $\boldsymbol{v_5(x,y,t)}$ when $\boldsymbol{\gamma = 0.7, 0.8, 0.9, 1}$

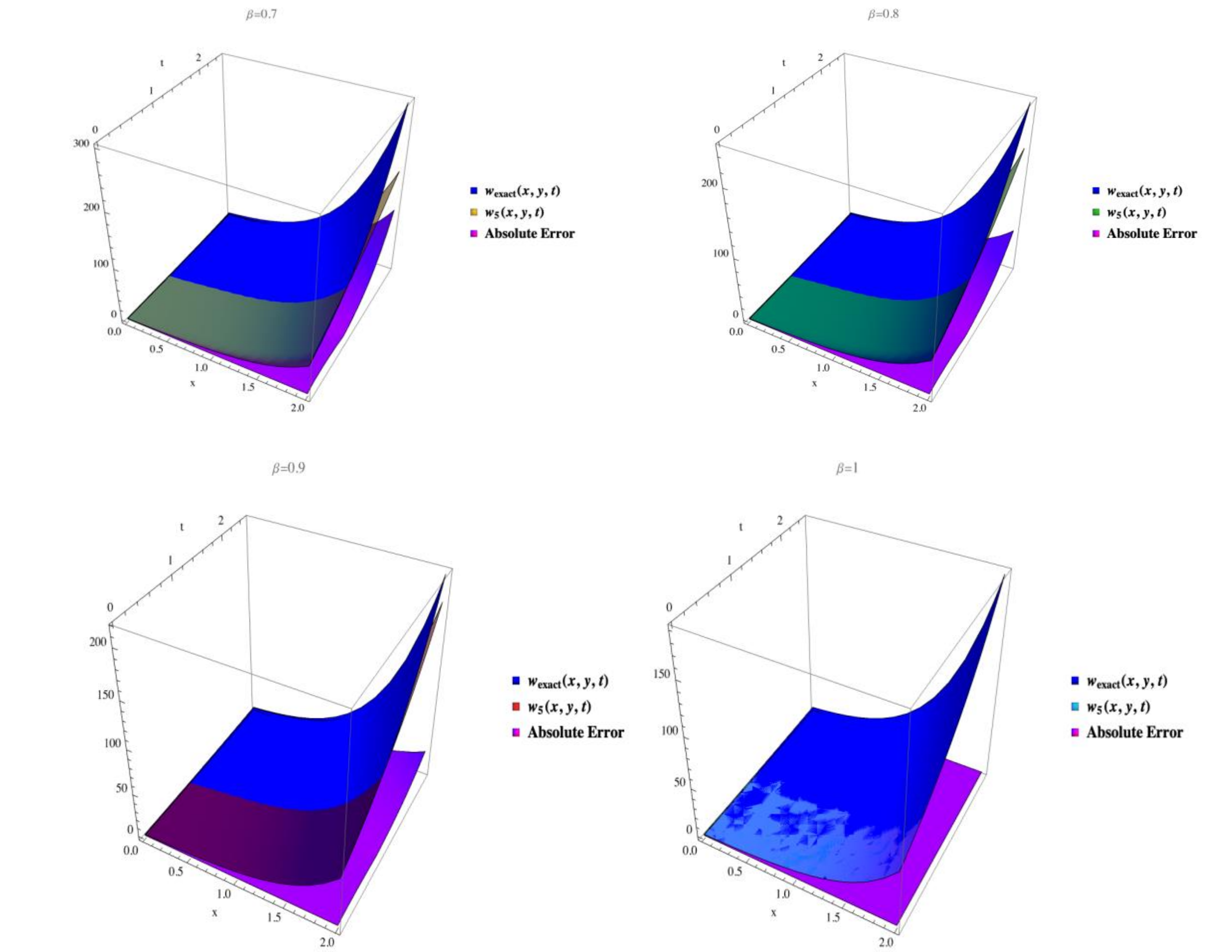

Figure 3: The 3D plot comparison surfaces of exact solutions (24) and $\boldsymbol{w_5(x,y,t)}$ when $\boldsymbol{\beta=\gamma=0.7, 0.8, 0.9, 1}$

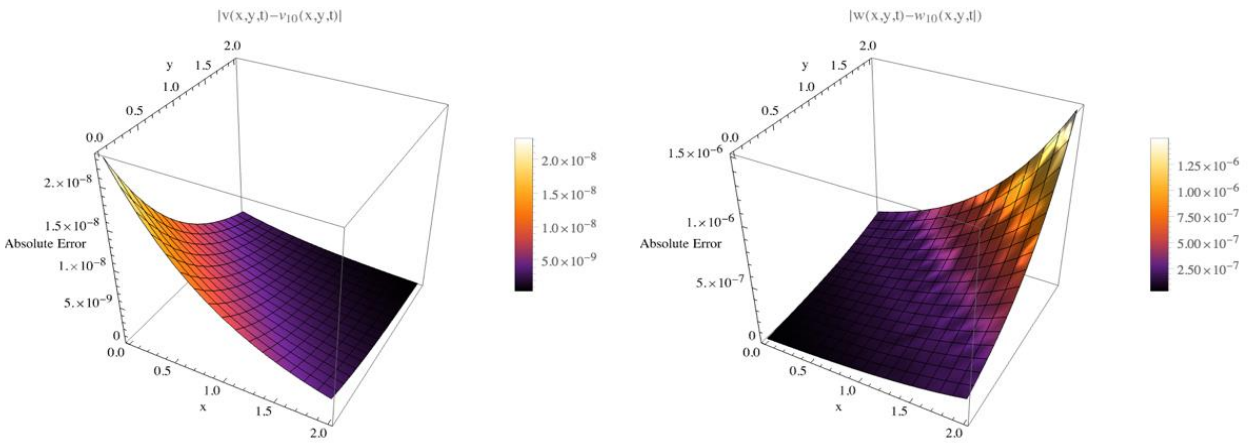

Figure 4: The absolute error of the 10th NDM approximate solutions when $\boldsymbol{\gamma=\beta=1}$.

Table 5: The comparison of the absolute errors $\boldsymbol{v_{10}(x,y,t)}$

	$x=y$	NDM	RDT [7]
$t=1$	0.3	6.44401×10^{-12}	6.44401×10^{-12}
	0.6	3.53656×10^{-12}	3.53662×10^{-12}
	0.9	1.94089×10^{-12}	1.94088×10^{-12}
	1.2	1.06519×10^{-12}	1.06519×10^{-12}
	1.5	5.84584×10^{-13}	5.84588×10^{-13}
$t=2$	0.3	1.26854×10^{-8}	1.26854×10^{-8}
	0.6	6.96188×10^{-9}	6.96188×10^{-9}
	0.9	3.82076×10^{-9}	3.82076×10^{-9}
	1.2	2.09688×10^{-9}	2.09688×10^{-9}
	1.5	1.15079×10^{-9}	1.15079×10^{-9}

Table 6: The comparison of the absolute errors $\boldsymbol{w_{10}(x,y,t)}$

	$x=y$	NDM	RDT[7]
$t=1$	0.3	2.32552×10^{-11}	2.32552×10^{-11}
	0.6	4.23732×10^{-11}	4.2375×10^{-11}
	0.9	7.72076×10^{-11}	7.72058×10^{-11}
	1.2	1.40684×10^{-10}	1.40684×10^{-10}
	1.5	2.56335×10^{-10}	2.56335×10^{-10}
$t=2$	0.3	4.97669×10^{-8}	4.9766910^{-8}
	0.6	9.06812×10^{-8}	9.0681210^{-8}
	0.9	1.65232×10^{-7}	1.65232×10^{-7}
	1.2	3.01072×10^{-7}	3.01072×10^{-7}
	1.5	5.48589×10^{-7}	5.48589×10^{-7}

Conclusions

The analytical solution of the two-dimensional nonlinear time-fractional reaction-diffusion Brusselator system in the Caputo sense was successfully obtained by implementing the fractional Natural decomposition method. We have demonstrated the reliability and trustworthiness of the method through numerical simulations over various values of the fractional order. In comparison with RDT, the NDM showed a great protentional in solving such two-dimensional initial value problems. Thus, we recommend for future investigation, the use of the method for solving either deterministic or stochastic higher multi-dimensional FPDEs as well as systems of FPDEs.

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