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RESEARCH ARTICLE **NUMERICAL ANALYSIS**

A Tunable One-Parameter Derivative-Free Secant-Type Method with Quadratic Convergence for Solving Nonlinear Equations

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ABSTRACT

Nonlinear equations arise across a wide range of scientific and applied disciplines, yet closed-form analytical solutions are often unattainable. The classical Secant Method remains a popular choice because it avoids derivative evaluation, although its superlinear convergence rate can be comparatively slow. To overcome this limitation, we introduce a modified secant-type method that retains the algorithm's original simplicity, requires only two initial approximations, and incorporates a single tunable parameter. Convergence analysis shows that the proposed method preserves superlinear convergence in general and attains quadratic convergence as the parameter approaches unity. Numerical experiments on a variety of nonlinear equations, tested under several initial approximations, demonstrate that the method converges efficiently and reliably to the exact root—in contrast to Newton's method and the three-point Secant method, both of which fail to converge in certain cases. Moreover, for equations with multiple roots, fixing the initial approximations while varying the tunable parameter causes the method to converge to different roots, underscoring the parameter's role in shaping the basins of attraction. By combining the derivative-free simplicity of the classical secant approach with convergence rates comparable to those of Newton's method and the three-point Secant method, the proposed technique offers a tunable, efficient, and reliable tool for solving nonlinear equations.

طريقة قابلة للضبط ذات معلمة واحدة، خالية من المشتقات، من نمط القاطع، ذات تقارب تربيعي لحل المعادلات غير الخطية

غادة اشتيوي [1,*]

الكلمات المفتاحية:

تعديل جديد
معادلات غير خطية
تقارب تربيعي
الجذور
طريقة القاطع

الملخص

تلعب المعادلات غير الخطية دورًا محوريًا في طيف واسع من المجالات العلمية والتطبيقية، غير أن التوصل إلى حلول تحليلية مغلقة الصيغة لها غالبًا ما يكون متعذرًا. وتظل الطريقة القاطعة التقليدية من الأساليب الشائعة الاستخدام، كونها لا تتطلب حساب المشتقة، إلا أن معدل تقاربها فوق الخطي قد يكون بطيئًا نسبيًا. ولتجاوز هذا القصور، نقدّم في هذا البحث طريقة معدّلة من نوع القاطعة تحافظ على بساطة الخوارزمية الأصلية، ولا تتطلب سوى تقريبين ابتدائيين، وتتضمن معلمة واحدة قابلة للضبط. ويُظهر تحليل التقارب أن الطريقة المقترحة تحافظ على التقارب فوق الخطي في الحالة العامة، وتحقق تقاربًا تربيعيًا عندما تؤول المعلمة إلى الواحد. كما تُظهر التجارب العددية، التي أُجريت على مجموعة متنوعة من المعادلات غير الخطية وباستخدام عدة تقريبات ابتدائية، أن الطريقة تتقارب بكفاءة وموثوقية نحو الجذر الدقيق، خلافًا لطريقة نيوتن والطريقة القاطعة ذات النقاط الثلاث، اللتين تفشلان في التقارب في بعض الحالات. علاوة على ذلك، بالنسبة للمعادلات ذات الجذور المتعددة، فإن تثبيت التقريبات الابتدائية مع تغيير قيمة المعلمة يؤدي إلى تقارب الطريقة نحو جذور مختلفة، مما يبرز دور هذه المعلمة في تشكيل أحواض الجذب. وبالجمع بين بساطة الطريقة القاطعة الخالية من المشتقة ومعدلات تقارب مماثلة لتلك التي تحققها طريقة نيوتن والطريقة القاطعة ذات النقاط الثلاث، توفر الطريقة المقترحة أداة قابلة للضبط وفعّالة وموثوقة لحل المعادلات غير الخطية.

Introduction

Nonlinear equations, which are frequently challenging or even impossible to solve analytically, are a common component of scientific problems. Analytical and numerical approaches have been widely employed to investigate mathematical problems arising in various scientific and engineering applications [1-3]. The secant method is one of the classical and efficient algorithms used to approximate the roots of a function, alongside other well-known methods such as the bisection method, the Newton–Raphson method, Müller's method, Brent's algorithm, and Steffensen's method, as discussed in numerous books and research articles [4–16]. Some of these algorithms require multiple function evaluations per iteration, and they typically require one or more initial estimations. The Newton–Raphson method, for example, requires evaluating the function and its derivative at every iteration [5,7–9,11,17,18]. This can be difficult or even impossible for many types of equations, especially when

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explicit derivatives are not available or their computation is more complicated than evaluating the original function. Because it only needs to evaluate the function at one point per iteration while retaining the previous function value at the other point, the secant method stands out for being straightforward and derivative-free [4,5,7,9,13].

The secant method has since been modified by several researchers with different aims [17,19–28]. Some modifications restore quadratic convergence by explicitly incorporating derivative evaluations [22,24], which sacrifices the derivative-free character that makes the secant method attractive in the first place. Others improve the convergence order by using three-point divided-difference formulas [27,29], at the cost of a more complex iterative expression, additional stored function values, and, in some cases, the absence of a rigorous proof of the convergence order. To date, no modification has been proposed that preserves both the two-point structure and the derivative-free nature of the classical secant method while still allowing the convergence order to be increased beyond superlinear.

In this paper, we address this gap by introducing a single tunable parameter α_k into a central-difference approximation of Newton's method, used to construct an auxiliary point that replaces x_{k+1} on the right-hand side of the iterative formula. Unlike previous derivative-based or three-point modifications, this construction requires only the same two data points as the classical secant method, keeping the per-iteration cost close to that of the classical scheme, while the parameter α_k provides direct control over the convergence order: the method retains superlinear convergence in general and attains quadratic convergence as $\alpha_k \to 1$. We prove this convergence result formally in Section 3, and we further show numerically that varying α_k can redirect convergence toward different roots in multiple-root problems, which highlights the role of the parameter in shaping the method's basins of attraction.

Finally, the proposed method was compared with the Newton–Raphson method and the three-point secant method [27]. All numerical experiments and graphical illustrations were carried out using MATLAB R2020a.

Secant Method Developmen

The proposed modification builds upon the classical Newton–Raphson method, which is defined as [5,7-9,11,23]:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0,1,2,\dots$$

We use a first-order central difference scheme to approximate the derivative because evaluating it is frequently challenging or computationally costly. The iterative formula results from this:

$$x_{k+1} = x_k - \frac{f(x_k)(x_{k+1} - x_{k-1})}{f(x_{k+1}) - f(x_{k-1})}$$

We substitute an auxiliary approximation $\tilde{x}_{k+1}$ for x_{k+1} on the right side, yielding

$$x_{k+1} = x_k - \frac{f(x_k)(\tilde{x}_{k+1} - x_{k-1})}{f(\tilde{x}_{k+1}) - f(x_{k-1})}, \quad k = 1,2,\dots$$

Where $\tilde{x}_{k+1} = x_k + \alpha_k(x_k - x_{k-1})$, the parameter α_k is chosen as a variable to enforce $x_{k+1} - x_k \approx \alpha_k(x_k - x_{k-1})$.

Convergence of the method

In a neighbourhood of an isolated zero p, let f be a real-valued function of a real variable x with n continuous derivatives. We take into consideration the following well-known iterative processes in order to create an approximating sequence $\{x_k\}$ that converges to the simple real root p:

$$x_{k+1} = x_k - \frac{f(x_k)(\tilde{x}_{k+1} - x_{k-1})}{f(\tilde{x}_{k+1}) - f(x_{k-1})} \qquad (1)$$

By defining the error at the k-th step as $e_k = x_k - p$, Equation (1) can be expressed as follows:

$$e_{k+1} = e_k - \frac{f(x_k)(\tilde{e}_{k+1} - e_{k-1})}{f(\tilde{x}_{k+1}) - f(x_{k-1})} \qquad (2)$$

Where $\tilde{e}_{k+1} = e_k + \alpha_k(e_k - e_{k-1})$. By Taylor's theorem around the real root p, we have

$$f(\tilde{x}_{k+1}) = \tilde{e}_{k+1}c_1 + \tilde{e}_{k+1}^2 c_2 + O(\tilde{e}_{k+1}^3) \qquad (3)$$

$$f(x_k) = e_k c_1 + e_k^2 c_2 + O(e_k^3) \qquad (4)$$

$$f(x_{k-1}) = e_{k-1}c_1 + e_{k-1}^2 c_2 + O(e_{k-1}^3) \qquad (5)$$

where $c_1 = f'(p)$, $c_2 = \frac{f''(p)}{2}$. From equations (2), (3), (4), and (5), we have

$$e_{k+1} = e_k - \frac{(\tilde{e}_{k+1} - e_{k-1})(e_k c_1 + e_k^2 c_2)}{(\tilde{e}_{k+1} - e_{k-1})(c_1 + (\tilde{e}_{k+1} + e_{k-1})c_2)}$$

The preceding equation is rearranged into the following form:

$$e_{k+1} = e_k\left(\frac{(\tilde{e}_{k+1} + e_{k-1})c_2 - e_k c_2}{c_1 + (\tilde{e}_{k+1} + e_{k-1})c_2}\right)$$

Since $(\tilde{e}_{k+1} + e_{k-1})c_2$ tends to zero, it can be disregarded. By substituting for $\tilde{e}_{k+1}$, the equation can then be written as

$$e_{k+1} = e_k\left(\frac{(\alpha_k e_k + (1-\alpha_k)e_{k-1})c_2}{c_1}\right) \qquad (6)$$

Assuming the convergence rate is r, there is a constant A such

$$e_{k+1} \approx A e_k^r \ \ and \ \ e_k \approx A e_{k-1}^r \ \ or \ \ e_{k-1} \approx \left(\frac{1}{A}e_k\right)^{\frac{1}{r}}$$

Consequently, the equation (6) becomes

$$e_k^r \approx \frac{\alpha_k c_2}{A c_1}e_k^2 + \frac{(1-\alpha_k)c_2}{A^{1+\frac{1}{r}}c_1}\ e_k^{1+\frac{1}{r}}$$

When α_k tends to one, we find

$$|e_k|^r \approx \left|\frac{c_2}{Ac_1}\right||e_k|^2 = \left|\frac{f''(p)}{2Af'(p)}\right||e_k|^2$$

Therefore, the convergence rate of the modified secant method is quadratic. Now If α_k does not tend to 1, and $(1-\alpha_k)$ remains nonzero asymptotically, the lower-order term dominates, and the convergence order is determined by $r = 1 + \frac{1}{r}$, which gives $r = \frac{1+\sqrt{5}}{2}$.

Remark: The choice of the parameter α_k involves a trade-off between convergence order and numerical stability. According to the error equation (6), as $\alpha_k \to 1$, the coefficient $1 - \alpha_k$ tends to zero, suppressing the lower-order error term and allowing the quadratic term to dominate. Hence, the limiting case $\alpha_k \to 1$ leads to quadratic convergence.

Although setting $\alpha_k = 1$directly gives the quadratic limiting case, it may cause the denominator in the iterative formula (1) to vanish, making the method undefined at that iteration. This situation occurs in Example 4, presented later, where the algorithm fails because the denominator becomes zero. To avoid this difficulty while remaining numerically close to the limiting case, we use $\alpha_k = 1 - 10^{-10}$. This value is adopted in most numerical experiments for simplicity and numerical stability. The obtained results exhibit behavior close to that of the limiting case $\alpha_k \to 1$. In some examples, a sequence α_k satisfying $\alpha_k \to 1\ \forall\, k$ is employed to improve the numerical performance and reduce the number of iterations, as demonstrated in the subsequent numerical experiments.

Numerical Examples

This section presents and applies the suggested modified method to a set of illustrative examples. The results are then systematically compared with those of the classical Newton method [5,7-9], as well as the three-point method [27]:

$$x_{k+1} = x_{k-2} - \frac{f(x_{k-2})\big(f(x_k) - f(x_{k-1})\big)}{\left(\frac{f(x_k) - f(x_{k-2})}{x_k - x_{k-2}}\right)\big(f(x_k) - f(x_{k-1})\big) - f(x_k)\left(\frac{f(x_k) - f(x_{k-2})}{x_k - x_{k-2}} - \frac{f(x_{k-1}) - f(x_{k-2})}{x_{k-1} - x_{k-2}}\right)}$$

The objective is to check the efficiency and accuracy of the developed approach in comparison with the established methods reported in the mathematical literature.

In all numerical experiments, the iteration process was terminated when the absolute residual satisfied $|f(x_k)| < 10^{-16}$.

Example 1: Let $f(x) = e^{-x} - x - \sin(x)$, which has an exact root at 0.354463104375025. Figure 1 depicts the graph of the function.

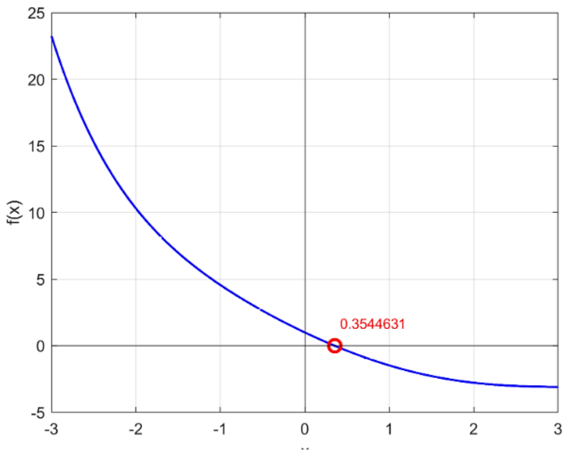

Figure 1: The graph of the $e^{-x} - x - \sin(x)$

It was tested using four different initial approximations to approximate the root by applying the new modified secant method, the three-point secant method, and the Newton method. For simplicity, the parameter in the new modified secant method was fixed to $\alpha_k = 1 - 10^{-10}$ for all k. The numerical results are shown in the following table 1.

Table 1: Comparative of different numerical methods for solving $e^{-x} - x - \sin(x)$ using several initial approximations

Initial approximations			Number of iterations required		
x_0	x_1	x_2	New modified secant	Three-point secant	Newton
0	0.5	1.5	4	5	4
1	1.5	2	6	6	5
-1	0	1	4	5	5
-12	-12.5	-13	18	15	17

In this example, the Newton's method and the new modified secant method both require four iterations when the initial approximation $x_0 = 0, x_1 = 0.5, x_2 = 1.5$ is selected. In contrast, the three-point secant method requires five iterations. The Newton's method converges at the fifth iteration for $x_0 = 1, x_1 = 1.5, x_2 = 2$, outperforming the three-point secant and the new modified secant methods by one iteration. In contrast to the Newton's method and the three-point secant method, which both converge at the fifth iteration, it is also obvious that the new modified Secant method reaches convergence after just four iterations when the initial approximations $x_0 = -1, x_1 = 0, x_2 = 1$ are used. However, when the initial approximations are chosen far from the root, all methods require a larger number of iterations to approximate the exact root; the new modified secant method terminates at the eighteenth iteration, the Newton's method converges at the seventeenth iteration, while the three-point secant method converges at the fifteenth iteration.

The CPU times were 0.1292 s for Newton's method, 0.00229 s for the new modified secant method, and 0.001155 s for the three-point secant method.

Example 2: The function $f(x) = x\cos(x) - 1$ has infinitely many roots, as shown in figure 2. Therefore, a set of initial approximations was selected in the vicinity of specific roots of the function. In the new modified secant method, we used $\alpha_k = 1 - 10^{-10}$ for all k, as shown in table 2.

This function has infinitely many roots; therefore, a set of initial approximations was selected in the vicinity of specific roots of the function. In the new modified secant method, we used $\alpha_k = 1 - 10^{-10}$ for all k,as shown in Table 2.

Table 2: Comparative different numerical methods for solving $x\cos(x) - 1$ using multiple initial approximations

Methods	**Initial approx.-imations**	**Number of iterations required**	**Converge to**
New modified secant	1.5, 1.75, 1.875	19	7.724153192
Three-point secant		Fail	-
Newton		Fail	-
New modified secant	-2.5, -2, -1.5	4	−2.073932809
Three-point secant		5	−2.073932809
Newton		4	−2.073932809
New modified secant	-5, -4, -3	5	−4.487669603
Three-point secant		7	−4.487669603
Newton		5	−4.487669603
New modified secant	−7, −6.5, −6	17	11.08590173
Three-point secant		12	−4.487669603
Newton		5	−7.979631071

The new modified secant method in this example converges at the nineteenth iteration to the root 7.724153192 when the initial approximations $x_0 = 1.5,\ x_1 = 1.75,\ and x_2 = 1.875$ are selected. In contrast, Newton's method and the three-point secant method fail and do not converge to any root of the function. The new modified Secant method and Newton's method both converge to the root -2.073932809 at the fifth iteration when the initial approximations $x_0 = -2.5,\ x_1 = -2,\ x_2 = -1.5$ are chosen. The three-point secant method also converges to the same root at the fifth iteration. Also, for

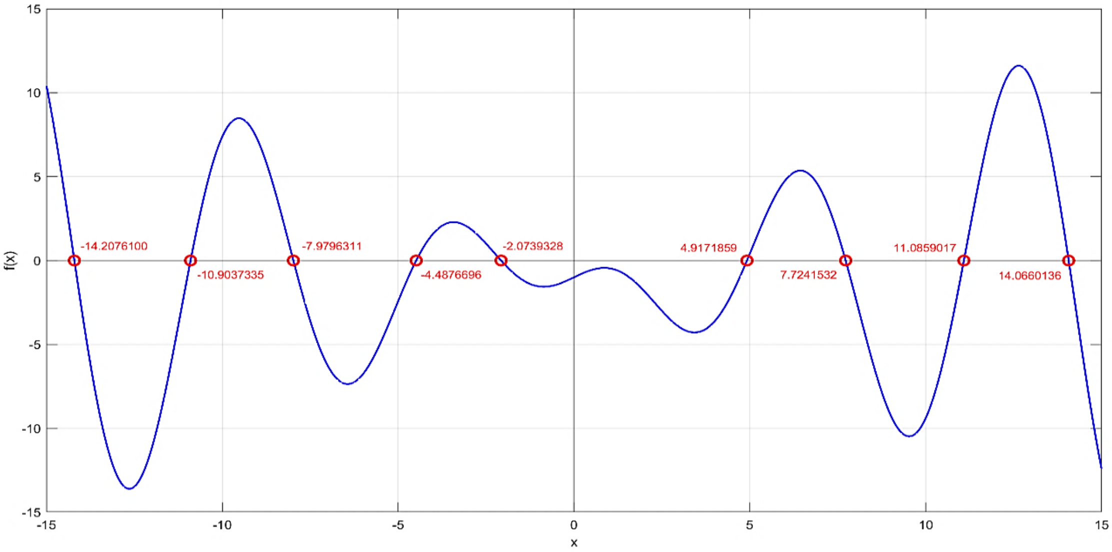

Figure 2: The graph of the $xcos(x) - 1$

the initial approximations $x_0 = -5,\ x_1 = -4,\ x_2 = -3$, Newton's method and the new modified secant method both converge to the root -4.487669603 at the fifth iteration, while the three-point secant method does so at the seventh. Finally, if the initial approximations are of the form $x_0 = -7,\ x_1 = -6.5,\ x_2 = -6$, the new modified secant method converges to the root 11.08590172877 in the seventeenth iteration, even though it is quite far from the initial approximations. In contrast, Newton's method converges to the root -7.979631071097 at the fifth iteration, while the three-point secant method reaches the root -4.487669603 at the twelfth iteration.
The CPU times were 0.245227 s for Newton's method, 0.0028189 s for the new modified secant method, and 0.0081486 s for the three-point secant method.

Remark: Consider the previous example with initial approximations $x_0 = -7, x_1 = -6.5, and\ x_2 = -6$. The behavior of the modified secant method is highly sensitive to the choice of the sequence α_k:
For $\alpha_0 = 0.1$, and $\alpha_k = \alpha_{k-1} + 9 \times 10^{-k}$, the method converges to the root -2.07393280909 at iteration 8.
For $\alpha_0 = 0.1$, and $\alpha_k = \alpha_{k-1} + 9 \times 10^{-(k+2)}$, the method instead reaches the root -7.97963107097 at iteration 8.
For $\alpha_k = 0.92 + \sum_{i=1}^{k-1} 9 \times 10^{-i}$ (k $\geq$ 1), the method converges to a different root, -4.86741398947, at iteration 13.
These results demonstrate that, for the modified secant method, both the rate of convergence and the particular root obtained depend critically on the choice of the parameter sequence α_k.

Example 3: The function $f(x) = \frac{1}{1+x^2} - x$ has roots at $x = 0.68232780382802$, As depicted in the following figure 3. Table 3 summarizes and presents the results obtained by using a set of initial approximations and applying the algorithms, with $\alpha_k = 1 - 10^{-10}$ for all k in the new modified secant method.

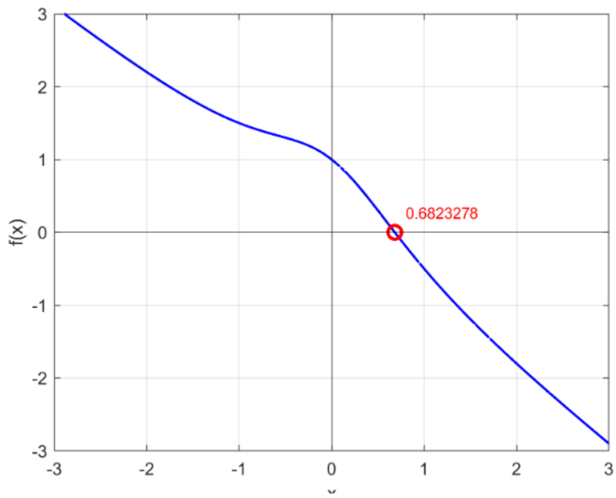

Figure 3: The graph of the $\frac{1}{1+x^2} - x$

Table 3: Comparative different numerical methods for solving $\frac{1}{1+x^2} - x$ using multiple initial approximations.

Initial approximations			**Number of iterations require**		
x_0	x_1	x_2	**New modified secant**	**Three-point secant**	**Newton**
-10	-7	-4	7	7	6
-1	1	3	5	5	6
0.2	0.4	0.6	5	5	5
5	8	11	7	7	7

From the results in table 3, it can be noticed that the three methods converge almost in an identical manner for all the chosen initial approximations, either close or far from the exact root.
The CPU times were 0.459 s for Newton's method, 0.000941 s for the new modified secant method, and 0.001267 s for the three-point secant method.

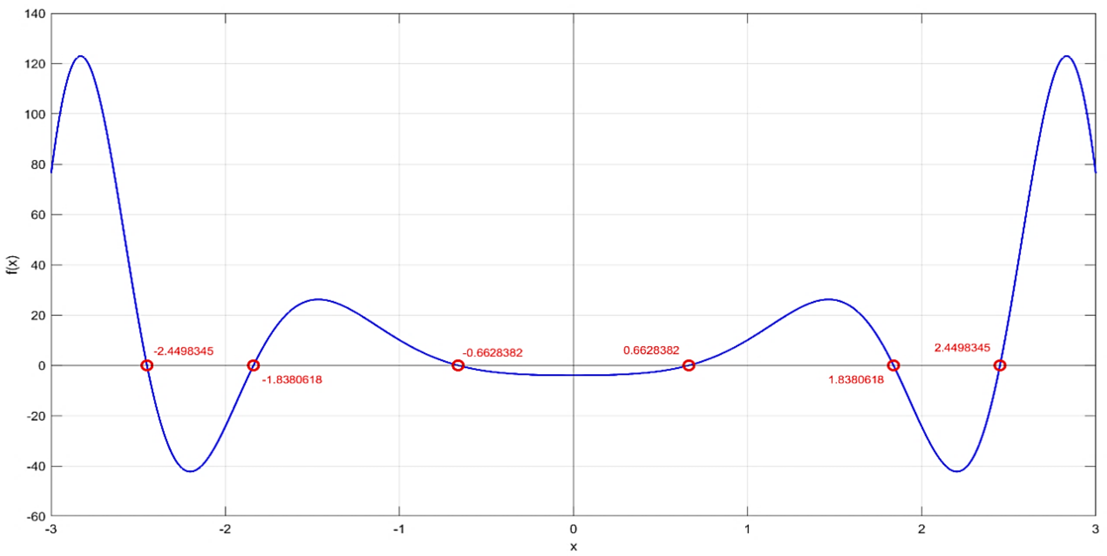

Figure 4: The graph of the $f(x) = 4x^2 + 12x^2\sin(x^2) - 4$

Example 4: The function $f(x) = 4x^2 + 12x^2\sin(x^2) - 4$ has infinite roots, as shown in figure 4.

The following table 4 shows the results obtained from applying the three algorithms to the equation under study, where the parameter $\alpha_k = 1 - 10^{-10}$ was fixed for each $k \geq 1$ in the new modified secant method in order to simplify the implementation procedure, except for the case with initial approximations $x_0 = -0.5, x_1 = 0, x_2 = 0.5$, where it was assumed that $\alpha_k = 1 - 10^{-5k+4}$ $\forall\, k \geq 1$ to ensure convergence.

Table 4: Comparative different numerical methods for solving $4x^2 + 12x^2\sin(x^2) - 4$ using multiple initial approximations.

Methods	Initial appro-ximations	Number of iterations required	Converge to
New modified secant $\boldsymbol{\alpha_k = 1 - 10^{-5k+4}}$	-0.5, 0, 0.5	10	-0.662838243
Three-point secant		Fail	-
Newton		6	0.662838243
New modified secant	0, 0.75, 1.5	4	0.662838243
Three-point secant		6	0.662838243
Newton		Div	–
New modified secant	1.5, 1.7, 1.9	5	1.838061774
Three-point secant		5	1.838061774
Newton		7	4.003226488

From the table 4, it is clear that the employed numerical methods showed a quite different behavior depending on the initial guesses. Indeed, for some of the obtained initial approximations, three-Point Secant methods converged, whereas the newton method converged to the root 0.662838243073 within 6 iterations and the New Modified Secant method converged to the -0.662838243073 in 10 iterations. Also, the new modified secant method also demonstrated rapid convergence (4–5 iterations) with the initial approximations $x_0 = 0$, $x_1 = 0.75$, $x_2 = 1.5$ and $x_0 = 1.5$, $x_1 = 1.7$, $x_2 = 1.9$. The three-point secant method produced similar results as that of the new modified secant method in terms of speed and accuracy, although it required slightly more iterations in some cases (5–6 iterations). On the other hand, Newton method diverged for certain initial approximations, resulting in divergence, the divergence is due to the fact that the derivative of the function is very small at the initial approximations.

The CPU times were 0.2881 s for Newton's method, 0.0011375 s for the new modified secant method, and 0.0011857 s for the three-point secant method.

Remark. When $\alpha_k = 1$ for all k, the denominator of the iterative formula (Eq. (1)) becomes zero at the first iteration for the initial approximations $x_0 = -0.5, x_1 = 0$. Consequently, the method fails and becomes undefined at that step.

Example 5: The function $f(x) = ln(x) - x + 1$ has roots at x = 1, As depicted in the figure 5.

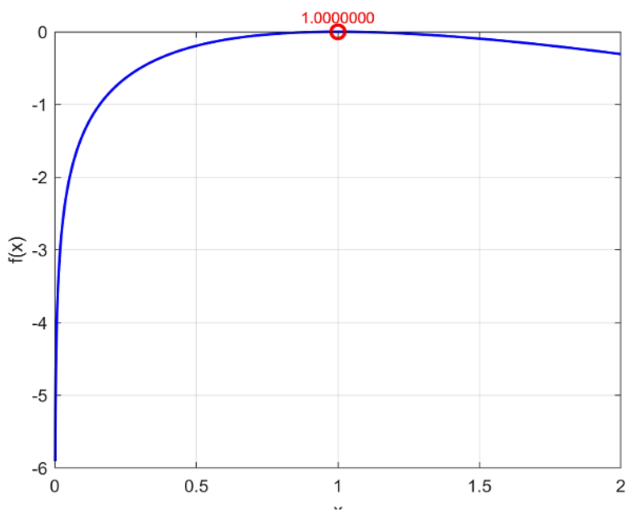

Figure 5: The graph of the $ln(x) - x + 1$

By applying the employed methods with the initial approximations $x_0 = 1.25, x_1 = 1.625, x_2 = 1.8125$ and $x_0 = 0.75, x_1 = 1.5, x_2 = 2$, where $\alpha_k = 1 - 10^{-10}$ for all k in the new modified secant method, the obtained results are presented in Table 5.

Table 5: Comparative different numerical methods for solving $ln(x) - x + 1$ using multiple initial approximations.

Methods	Initial approximations	Number of iterations required	Converge to
New modified secant	1.25, 1.625, 1.8125	25	1.000000008863
Three-point secant		29	1.00000005037
Newton		24	1.00000001387
New modified secant	0.75, 1.5, 2	24	1.000000007165
Three-point secant		25	1.0000003102678
Newton		25	9.999999930776

The results obtained show that the studied methods converge to the exact root with good accuracy; however, they require a relatively large number of iterations, from 24 to 29 for all the initial approximations. Also, a slight superiority is observed for the new modified secant method regarding the number of iterations and accuracy of approximation.
The CPU times were 0.667945 s for Newton's method, 0.0009445 s for the new modified secant method, and 0.0030236 s for the three-point secant method.
Example 6: Consider the function $f(x) = x^4 + 2x^2 - x - 3$. This function has two real roots $x = 1.124123029704, -0.876053115817$ and two complex roots $x = -0.12403496 \pm 1.7409611i$. Figure 6 illustrates the real roots of the function.

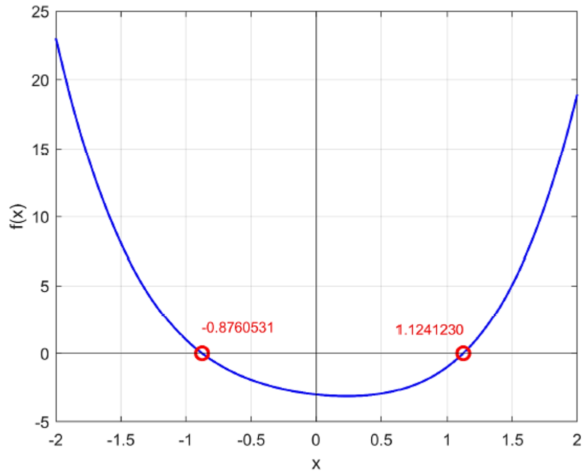

Figure 6: The graph of the $x^4 + 2x^2 - x - 3$

By choosing the different initial approximations and using the three algorithms with $\alpha_k = 1 - 10^{-10}$ for all k in the new modified secant method given above, following table 6 presents the obtained results.

Table 6: Comparative different numerical methods for solving $x^4 + 2x^2 - x - 3$ using multiple initial approximations.

Initial approximations			**Number of iterations require**			**Converge to**
x_0	x_1	x_2	**New modified secant**	**Three-point secant**	**Newton**	
1.5	2	2.5	7	6	5	1.1241230297
0.5	1	1.5	4	5	8	1.1241230297
-2	-1.5	-1	6	6	7	-0.8760531158
-0.1	0.1	0.2	12	9	8	-0.8760531158
-8	-6	-4	11	10	11	-0.8760531158

Based on the previous results, we observe that, with differences ranging from one to four iterations, all methods converge to the same root for all selected initial approximations. For the initial approximations $x_0 = 1.5, x_1 = 2$ and $x_2 = 2.5$, Newton's method is superior to the three-point secant method by one iteration, which performs one iteration better than the new modified secant method. However, for the initial approximations $x_0 = 0.5, x_1 = 1$ and $x_2 = 1.5$, the new modified secant method performs better than the three-point secant method by one iteration, while Newton's method requires twice as many iterations as the new modified secant method. Furthermore, in the last initial approximation, the three-point secant method converges to the root at iteration 10, whereas both the new modified secant method and Newton's method converge at iteration 11. While, if $x_0 = -6$ and $x_1 = -4$ are chosen, then both methods converge at the same iteration, 10.
The CPU times were 0.1991527 s for Newton's method, 0.0014258 s for the new modified secant method, and 0.0036679 s for the three-point secant method.
Example 7: The function $f(x) = \sqrt{x^2 + 1} - 2$ has real roots at $x = -\sqrt{3}$ and $x = \sqrt{3}$, as shown in Figure 7.

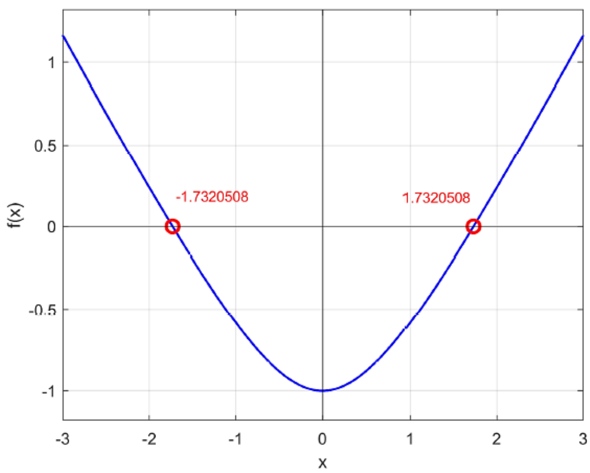

Figure 7: The graph of the $f(x) = \sqrt{x^2 + 1} - 2$

The results presented in Table 7 were obtained by applying the three previously mentioned algorithms to this function using various initial approximations, with $\alpha_k = 1 - 10^{-10}$ for all k in the new modified secant method.

Table 7: Comparative different numerical methods for solving $\sqrt{x^2+1}-2$ using multiple initial approximations

Methods	Initial approx-imations	Number of iterations required	$f(x_n)$	Converge to
New modified secant	-5, -4.4, -4	5	0	-1.73205081
Three-point secant		7	0	-1.73205081
Newton		4	0	-1.73205081
New modified secant	-2, -1.5, -1	4	0	-1.73205081
Three-point secant		4	0	-1.73205081
Newton		4	$4.441E-16$	-1.73205081
New modified secant	1, 2, 3	4	0	1.73205081
Three-point secant		4	0	1.73205081
Newton		4	0	1.73205081
New modified secant	10, 11, 12	6	0	1.73205081
Three-point secant		7	$7.382E-09$	1.73205081
Newton		4	$4.885E-15$	1.73205081

Regardless of the initial approximations selected, the three numerical methods all converge to the same root, according to the last table's results. In the first approximations, both Newton's method and the new modified secant method exhibited comparable convergence, differing by only one iteration, while the three-point secant method required seven iterations to reach the solution. For the second and third initial approximations, the three methods showed identical behavior in terms of the number of iterations needed for convergence. In the final initial approximations, Newton's method demonstrated the fastest convergence compared to both the modified secant and the three-point secant methods. However, the three-point secant method's approximate root accuracy did not exceed the order of magnitude 10^{-9}, making it less precise than the results achieved by Newton's method and the new modified secant method.

The CPU times were 0.31782 s for Newton's method, 0.00087662 s for the new modified secant method, and 0.001134 s for the three-point secant method.

Comparison: when $\alpha_k = 1$ was fixed for all k, essentially the same results were obtained in all numerical examples, with two exceptions: Example 4 with $x_0 = -0.5, x_1 = 0$, as discussed previously, and Example 7, where the method required 5 iterations instead of 4 for both sets of initial approximations, $x_0 = -2, x_1 = -1.5$ and $x_0 = 1, x_1 = 2$.

Conclusion

The obtained results in this study indicate that the new modified secant method provides an efficient and simplified approach for solving nonlinear equations. By introducing a new parameter that allows quadratic convergence as it approaches unity and superlinear convergence otherwise, the proposed method preserves the simplicity of the classical secant method while offering additional flexibility.

The numerical results demonstrate the flexibility and robustness of the proposed method, showing competitive performance with Newton's method and, in many cases, superior performance to the three-point secant method in terms of convergence speed and the number of iterations required. Moreover, the parameter plays an important role in determining both the computed root and the convergence rate, particularly for functions with multiple roots, providing an additional means of influencing the convergence path.

Overall, the proposed modified secant method combines ease of implementation, a derivative-free formulation, and a fast convergence rate. In the numerical experiments considered in this study, no divergence or failure to locate the desired roots was observed. However, as is generally the case with iterative methods, a greater number of iterations may be required when the initial approximations are far from the desired root.

Recommendations

Further research could extend the proposed method to systems of nonlinear equations, with particular emphasis on investigating its convergence behavior and computational performance in such systems.

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