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Approximate Analytical Solutions of the Time-Fractional Chen–Lee–Liu Equation Using the Asymptotic Homotopy Perturbation Method

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ABSTRACT

The time-fractional Chen–Lee–Liu equation is considered to be a special case of the time-fractional derivative nonlinear Schrödinger equation, which have found wide applications in nonlinear optics, particularly in the study of pulse propagation within optical fibers. This study investigates approximate analytical solutions of the time-fractional Chen–Lee-Liu equation using the Asymptotic Homotopy Perturbation Method (AHPM), with the time-fractional derivative defined in the Caputo sense. The proposed approach transforms the nonlinear fractional partial differential equation into a sequence of recursive problems whose solutions are constructed iteratively. The convergence-control parameters are determined through a least-squares minimization of the residual over the prescribed computational domain. The accuracy of the resulting truncated series is assessed by comparison with the exact solution available for the corresponding integer-order case. Numerical results, including absolute errors and residual norms for different fractional orders and truncation levels, demonstrate the convergence behaviour of the proposed approximation. The results indicate that AHPM can provide accurate approximate solutions for the considered time-fractional Chen-Lee-Liu model.

الحلول التحليلية التقريبية لمعادلة تشين-لي- ليو ذات الرتبة الكسرية الزمنية باستخدام طريقة الاضطراب الهوموتوبي التقاربي

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الكلمات المفتاحية

معادلة تشين-لي-ليو ذات الرتبة الكسرية الزمنية
طريقة الاضطراب الهوموتوبي التقاربي
المعادلات التفاضلية الجزئية غير الخطية الكسرية
تقليل الباقي
طريقة المربعات الصغرى.

الملخص

تعد معادلة تشين-لي-ليو ذات الرتبة الكسرية الزمنية تعميماً لمعادلة تشين-لي-ليو الكلاسيكية، حيث يتم إضافة المشتقة الكسرية في البعد الزمني للمعادلة وذلك لتفسير الأنظمة الفيزيائية المعتمدة على الذاكرة والتأثيرات غير المحلية. تعد هذه المعادلة أحد الحالات الخاصة لمعادلة شرودنجر غير الخطية ذات الرتبة الكسرية الزمنية والتي تستخدم على نطاق واسع في مجال البصريات غير الخطية. هذه الورقة البحثية تهدف الى إيجاد حل تقريبي تحليلي لمعادلة تشين-لي-ليو ذات الرتبة الكسرية الزمنية باستخدام طريقة الاضطراب الهوموتوبي التقاربي، حيث تم توصيف المشتقات الكسرية وفقاً لتعريف كابوتو. يحوّل النهج المقترح المعادلة التفاضلية الجزئية الكسرية غير الخطية إلى سلسلة من المسائل التكرارية الخطية، حيث يتم بناء حلولها بصورة تكرارية. ويتم تحديد معاملات التحكم في التقارب من خلال تصغير دالة المتبقي باستخدام طريقة المربعات الصغرى على المجال الحسابي المحدد. وتُقيَّم دقة المتسلسلة المقطوعة الناتجة من خلال مقارنتها بالحل الدقيق المتاح للحالة ذات الرتبة الصحيحة (غير الكسرية) المقابلة. كما تُظهر النتائج العددية، بما في ذلك الأخطاء المطلقة ومعايير المتبقي لمختلف الرتب الكسرية ومستويات القطع، سلوك تقارب التقريب المقترح. وتشير النتائج إلى أن طريقة الاضطراب الهوموتوبي التقاربي يمكن أن توفر حلولًا تقريبية دقيقة لنموذج لمعادلة تشين-لي-ليو الزمني الكسري المدروس.

Introduction

Nonlinear Fractional Partial Differential Equations (NFPDEs) generalize classical nonlinear partial differential equations by allowing derivatives of non-integer order in time, space, or both. They are used to model systems exhibiting memory, and nonlocal effects, in physics, biology, and engineering [1]. NFPDEs are important models that play vital roles in the field of nonlinear sciences including the fluid dynamics, nonlinear optics, plasma physics, biological systems, and quantum mechanics [2]. Among these NFPDEs, the Time-Fractional Derivative Nonlinear Schrödinger Equation (TFDNSE) has been very commonly studied in nonlinear

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optics. The Kaup–Newell, Chen–Lee–Liu, and Gerdjikov–Ivanov equations arise as distinct integrable reductions associated with nonlinear Schrödinger-type models [3]. The Time-Fractional Chen–Lee–Liu Equation (TFCLLE) which alternatively referred to as DNLSE-II model is utilized for studying soliton propagation through optical fibres is given by,

$$i\frac{\partial^\alpha u}{\partial t^\alpha}+p\frac{\partial^2 u}{\partial x^2}+q|u|^2u+ik|u|^2\frac{\partial u}{\partial x}=0. \qquad (1)$$

The function $u=u(x,t)$ is the complex-valued wave function, and the fractional derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ is taken in the Caputo sense. In the optical fiber context, the real constant p represents the group velocity of the wave packet parameter and the parameter q accounts for the Kerr nonlinearity, while the term related to k is usually associated with the self-steepening phenomena. In the optical-fiber setting, the coordinates x and t denote propagation distance and retarded time, respectively [4]. Also the equation has been utilized in the context of nonlinear transmission networks to investigate the transmission of solitary wave signal through a one-dimensional transmission network [3]. The time-fractional form better incorporates memory effects into the model. This generalization enriches the mathematical framework and expands the physical applicability of the TFCLLE [5].

Although several analytical, semi-analytical, and numerical techniques have been applied to Chen-Lee-Liu-type equations, including symmetry reductions [5], Adomian-based decomposition methods [4, 5, 7], tanh-coth method [6], and spectral collocation schemes [8], comparatively less attention has been devoted to the systematic use of AHPM combined with residual-based least squares optimization for the time-fractional Chen-Lee-Liu equation. This motivates the present study, which focuses on constructing an explicit recursive approximation and quantitatively assessing its convergence and accuracy.

In this work, we employ the Asymptotic Homotopy Perturbation Method (AHPM) to construct approximate solutions of the time-fractional Chen–Lee–Liu equation. The method converts the original nonlinear fractional problem into a sequence of recursive subproblems. The convergence-control parameters are subsequently determined by minimizing the residual in the least-squares sense. The resulting approximation is evaluated through comparison with the exact integer-order solution and by means of quantitative error measures [9, 10]. All the numerical computations in this paper were implemented in MATLAB version R2014b.

Background Concepts

Definition 1.1 [11]. A real function $f:(0,\infty)\to\mathbb{R}$ is said to be in the space C_μ ; $\mu\in\mathbb{R}$ if there is a real number $p>\mu$ such that $f(t)=t^pf_1(t)$ where $f_1(t)\in C(0,\infty)$ and it is said to be in the space C_μ^n ; $\mu\in\mathbb{R}, n\in\mathbb{N}$ if $f^{(n)}\in C_\mu$

Definition 1.2 [12] If $f\in C_\mu$, $\mu\geq -1$ and $\alpha\geq 0$, then the left-handed (left-sided) Riemann–Liouville fractional integral is given by,

$$(I_{a+}^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \qquad (2)$$

where $t\in(a,\infty); a>0$, and $\Gamma(.)$ is the Gamma function.

Definition 1.3 [12] If $f\in C_{-1}^n$, then the left-sided Caputo fractional derivative of f of order α where $n-1<\alpha\leq n$, $n\in\mathbb{N}$, is given by:

$$({}_{a+}^{C}D_t^\alpha f)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\frac{d^n}{ds^n}f(s)\,ds, \qquad (3)$$

where, $t\in(a,\infty); a>0$.

The Riemann–Liouville fractional integral and the Caputo fractional derivative have some properties that are shown below,

1- Linearity: Let $f,g\in C_\mu$, $\mu\geq -1$ and $\beta,\gamma\in\mathbb{C}$,then

$$I_{a+}^\alpha(\beta f+\gamma g)=\beta I_{a+}^\alpha f+\gamma I_{a+}^\alpha g. \qquad (4)$$

2- Left-handed Riemann–Liouville fractional integral of power function: Let $\alpha>0,\beta>0$ then,

$$I_{0+}^\alpha\left(\frac{t^\beta}{\Gamma(\beta\alpha+1)}\right)=\frac{t^{(\beta+1)\alpha}}{\Gamma((\beta+1)\alpha+1)}. \qquad (5)$$

3- Inverse with fractional integral (up to initial terms): Let $f\in C_{-1}^n$ and $n-1<\alpha\leq n$, $n\in\mathbb{N}$ then,

$$I_{a+}{}^\alpha({}_{a+}^{C}D_t^\alpha f)(t)=f(t)-\sum_{k=0}^{n-1}\frac{f^k(a)(t-a)^k}{k!}. \qquad (6)$$

Formulation of the Asymptotic Homotopy Perturbation Method.

The AHPM should be distinguished from the classical Homotopy Perturbation Method (HPM), Homotopy Analysis Method (HAM), and Optimal Homotopy Asymptotic Method (OHAM). In the present formulation, the embedding parameter generates a recursive perturbation expansion, while auxiliary/convergence-control functions or parameters are selected to improve the approximation through residual minimization. The notation and construction adopted below follow the formulation introduced in [13] and should be used consistently throughout the manuscript [13]. Consider the following NFPDE,

$$\frac{\partial^\alpha u}{\partial t^\alpha}+L\big(u(x,t)\big)+N\big(u(x,t)\big)+f(x,t)=0, \qquad (7)$$

subject to these initial conditions:

$$u^{(k)}(x,0)=g_k(x), k=0,1,2,\dots,n-1 ,$$

where the operator $\frac{\partial^\alpha}{\partial t^\alpha}$; $n-1<\alpha\leq n$ denotes the Caputo fractional derivative operator, L and N are linear and nonlinear operators, respectively. The function $u(x,t)$ is the unknown solution defined on D, and x and t denote spatial and temporal variables, respectively.

The homotopy $U: D\times[0,1]\to\mathbb{C}$ is constructed such that,

$$\frac{\partial^\alpha U}{\partial t^\alpha}+\lambda\big(L(U)+N(U)\big)+f(x,t)=0, \qquad (8)$$

where $\lambda\epsilon[0,1]$ is the embedding parameter. If $\lambda=1$, then

$$U(x;t;1)=u(x,t).$$

Thus, as λ approaches 1, the approximate solution $U(x,t;\lambda)$ approaches the exact solution $u(x,t)$ of equation (7).

Expanding $U(x,t;\lambda)$ in a Maclaurin series with respect to λ, leads to,

$$U(x,t;\lambda)=\sum_{m=0}^{\infty}u_m(x,t;b_i)\lambda^m, \qquad (9)$$

where,

$$u_m(x,t)=\frac{1}{m!}\frac{\partial^m U(x,t;\lambda)}{\partial\lambda^m}\bigg|_{\lambda=0}.$$

Furthermore, assume that $L\big(U(x,t;\lambda)\big)$ and $N(U(x,t;\lambda))$ are expressed as,

$$L\big(U(x,t;\lambda)\big)=-\sum_{i=0}^{\infty}\left(\sum_{m=0}^{i}B_{i+1-m}L_m\right)\lambda^i, \qquad (10)$$

where,

$$L_m=L(u_m), m=0,1,2,\dots$$

Also,

$$N\big(U(x,t;\lambda)\big) = -\sum_{i=0}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m}\, N_m\right)\lambda^i, \qquad (11)$$

where,

$$N_m = \frac{1}{m!}\frac{\partial^m N\big(U(x,t;\lambda)\big)}{\partial\lambda^m}\bigg|_{\lambda=0},$$

are called the He''s polynomials, and $B_m = B_m(x,t;b_i)$ are arbitrary auxiliary functions. Substituting equations (9), (10) and (11) into equation (8), leads to

$$\sum_{i=0}^{\infty}\frac{\partial^\alpha}{\partial t^\alpha}u_i\lambda^i - \lambda\sum_{i=0}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m}\, N_m\right)\lambda^i - \lambda\sum_{i=0}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m} L_m\right)\lambda^i + f = 0. \qquad (12)$$

By equating coefficients of like powers of λ^0 on both sides of equation (12), we obtain,

$$\frac{\partial^\alpha}{\partial t^\alpha}u_0(x,t;b_i) + f(x,t) = 0;\ u_0^{(k)}(x,0) = g_k(x),$$

First-order deformation problem has the following form,

$$\frac{\partial^\alpha}{\partial t^\alpha}u_1(x,t;b_i) = B_1(L_0+N_0);\ u_1^{(k)}(x,0) = 0.$$

Second-order deformation problem is given by,

$$\frac{\partial^\alpha}{\partial t^\alpha}u_2(x,t;b_i) = B_2(L_0+N_0) + B_1(L_1+N_1);\ u_2^{(k)}(x,0) = 0,$$

where $k = 0,1,2,\ldots,n-1$. In general,

$$\frac{\partial^\alpha}{\partial t^\alpha}u_m(x,t;b_i) = B_m(L_0+N_0) + B_{m-1}(L_1+N_1) + \cdots + B_1(L_{m-1}+N_{m-1}) = \sum_{j=0}^{m-1} B_{m-j}\big(L_j+N_j\big);\ u_m^{(k)}(x,0) = 0,$$

By solving the recursive equations iteratively we obtain the series solution,

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t;b_i), \qquad (13)$$

provided that the series converges. The convergence of the series solution to the exact solution depends critically on the choice of the auxiliary functions $B_m(x,t;b_i)$. In practice, an approximate solution can be obtained by truncating the series at order n,

$$\tilde{u}_n(x,t;b_i) = \sum_{m=0}^{n} u_m(x,t;b_i). \qquad (14)$$

By substituting equation (14) into equation (7), we obtain the residual function

$$R(x,t,b_i) = \frac{\partial^\alpha \tilde{u}_n(x,t;b_i)}{\partial t^\alpha} + L(\tilde{u}_n(x,t;b_i)) + N(\tilde{u}_n(x,t;b_i)) + f(x,t).$$

The parameters $b_i; i = 1,2,3,\ldots$ are then determined by minimizing the integral of the square of the residual using the least-squares procedure, ensuring that the approximate solution is close to the exact solution.

The construction of the AHPM shows that it is computationally appealing since the k^{th}-order deformation problem in the AHPM is formulated in a simpler manner. Nevertheless, the principal drawback of AHPM lies in the necessity of solving a system of nonlinear algebraic equations for the unknown convergence-control parameters b_i, $i = 1,2,3\ldots$. This limitation is shared with the optimal homotopy analysis method as a common computational challenge inherited in both methods.

The sufficient condition for the convergence of the series (13) is provided in [14] .

Theorem 2.1 If the solution components $u_m \in (X, \|.\|); m = 0,1,2,3,\ldots$ belongs to a Banach space such that $\exists 0 \le \gamma < 1$ such that $\| u_{m+1}\| \le \gamma \| u_m\|, \forall m \ge m_0$, for some $m_0 \in \mathbb{N}$, then the series of the solution converge.

Proof: : Consider the sequence of partial sums $\{s_n\}_{n\in\mathbb{N}}$ defined by,

$$s_n = \sum_{m=0}^{n} u_m$$

Since $(X, \|.\|)$ is a vector space, it is closed under addition, therefore, $s_n \in (X, \|.\|) \forall\, n \in \mathbb{N}$. Moreover, for all $n \ge m_0$,

$$\| s_{n+1} - s_n\| = \| u_{n+1}\| \le \gamma \| u_n\| \le \gamma^2 \| u_{n-1}\| \ldots < \gamma^{n-m_0+1}\| u_{m0}\|$$

As a result, we have for all $n \ge m \ge m_0$

$$\begin{aligned}\| s_n - s_m\| &= \|(s_n - s_{n-1}) + (s_{n-1} - s_{n-2}) + \cdots + (s_{m+1} - s_m)\| \\ &\le \| s_n - s_{n-1}\| + \| s_{n-1} - s_{n-2}\| + \cdots + \| s_{m+1} - s_m\| \\ &\le \gamma^{n-m_0}\| u_{m0}\| + \gamma^{n-m_0-1}\| u_{m0}\| + \cdots + \gamma^{m-m_0+1}\| u_{m0}\| \\ &= \gamma^{m-m_0}(\gamma^{n-m} + \gamma^{n-m-1} + \gamma^{n-m-2} + \cdots + \gamma^1)\| u_{m0}\| \\ &= \gamma^{m-m_0}\| u_{m0}\| \sum_{j=1}^{n-m}\gamma^j = \gamma^{m-m_0}\| u_{m0}\| \frac{\gamma(1-\gamma^{n-m})}{1-\gamma} \\ &= \| u_{m0}\| \frac{(\gamma^{m-m_0+1} - \gamma^{n-m_0+1})}{1-\gamma}\end{aligned}$$

Since $\| u_{m0}\| < \infty$ we have,

$$\lim_{n,m\to\infty}\| s_n - s_m\| = 0$$

Therefore $\{s_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in a Banach space $(X, \|.\|)$, thus it converges to a point $s \in (X, \|.\|)$. As a result the series,

$$\sum_{m=0}^{\infty} u_m = \lim_{n\to\infty}\sum_{m=0}^{n} u_m = \lim_{n\to\infty} s_n = s,$$

converges to s.

Numerical Results and Discussion

In this section, we report the numerical outcomes obtained by employing the AHPM to solve the TFCLLE. The iterative decomposition process yields approximate solutions whose accuracy is carefully examined. Comparisons with analytical solutions illustrate the efficiency of AHPM in handling the equation of this type. A critical discussion highlights the numerical behaviour, and practical relevance of the obtained solutions.

Example 1. Consider the following TFCLLE:

$$i\frac{\partial^\alpha u}{\partial t^\alpha} + p\frac{\partial^2 u}{\partial x^2} + q|u|^2 u + ik|u|^2\frac{\partial u}{\partial x} = 0, \qquad (15)$$

under the initial condition,

$$u(x,0) = e^{icx},$$

where $(x,t) \in D = [-2,2] \times [0,1]$. The exact solution for equation (15) at integer order $\alpha = 1$ is given as,

$$u(x,t) = e^{icx - i(c^2p - q + ck)t}. \qquad (16)$$

Solution: We can rewrite the equation (15) as,

$$\frac{\partial^\alpha u}{\partial t^\alpha} - ip\frac{\partial^2 u}{\partial x^2} - iq|u|^2 u + k|u|^2\frac{\partial u}{\partial x} = 0.$$

The linear and the nonlinear part of this equation are given by:

$$L\big(u(x,t)\big) = -ip\frac{\partial^2 u(x,t)}{\partial x^2},$$

$$N(u(x,t)) = -iq|u|^2 u + k|u|^2\frac{\partial u}{\partial x}.$$

The following homotopy function $U: D\times[0,1]\to\mathbb{C}$ is constructed as follows,

$$\frac{\partial^\alpha U}{\partial t^\alpha} + \lambda\big(L(U) + N(U)\big) = 0. \qquad (17)$$

Now we assume that,

$$U(x,t;\lambda) = \sum_{i=0}^{\infty} u_i(x,t)\lambda^i. \qquad (18)$$

Then, by applying the following initial condition:

$$U(x,0;\lambda) = e^{icx} = \sum_{i=0}^{\infty} u_i(x,0)\lambda^i,$$

$$u_0(x,0) = e^{icx},$$

$$u_i(x,0) = 0;\ i = 1,2,3,\ldots$$

We can rewrite the operators L and N as follows,

$$L = -\sum_{i=0}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m}L_m\right)\lambda^i;\ L_m = L(u_m), \qquad (19)$$

$$N = -\sum_{i=0}^{\infty}\left(\sum_{m=0}^{i} B_{i+1-m}N_m\right)\lambda^i, \qquad (20)$$

for $m = 0,1,2,\ldots$

By substituting equations (18), (19) and (20) into equation (17), then comparing the coefficients of λ^iwe get,

$$\lambda^0: \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0 \quad ;\ u_0(x,0) = e^{icx},$$

$$\lambda^1: \frac{\partial^\alpha u_1}{\partial t^\alpha} = B_1(L_0 + N_0) \quad ;\ u_1(x,0) = 0,$$

$$\lambda^2: \frac{\partial^\alpha u_2}{\partial t^\alpha} = B_2(L_0 + N_0) + B_1(L_1 + N_1) \quad ;\ u_2(x,0) = 0,$$

$$\lambda^3: \frac{\partial^\alpha u_3}{\partial t^\alpha} = B_3(L_0 + N_0) + B_2(L_1 + N_1) + B_1(L_2 + N_2)\ ;\ u_3(x,0) = 0.$$

In general,

$$\lambda^m: \frac{\partial^\alpha}{\partial t^\alpha}u_m(x,t;b_i) = \sum_{j=0}^{m-1} B_{m-j}\big(L_j + N_j\big);\ u_m(x,0) = 0,$$

The He''s polynomials for the following nonlinear part,

$$N(u(x,t)) = -iq|u|^2 u + k|u|^2\frac{\partial u}{\partial x} = -iq|u|^2 u + k|u|^2 u_x$$

are given by,

$$N_0 = \frac{1}{0!}N\left(\sum_{i=0}^{\infty} u_i\lambda^i\right)\Bigg|_{\lambda=0} = -iqu_0^2\overline{u_0} + ku_0u_{0_x}\overline{u_0},$$

$$N_1 = \frac{1}{1!}\frac{\partial^1}{\partial\lambda^1}N\left(\sum_{i=0}^{\infty} u_i\lambda^i\right)\Bigg|_{\lambda=0} = -2iqu_0u_1\overline{u_0} - iqu_0^2\overline{u_1} + ku_{0_x}u_1\overline{u_0} + ku_0u_{1_x}\overline{u_0} + ku_0u_{0_x}\overline{u_1},$$

$$N_2 = \frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}N\left(\sum_{i=0}^{\infty} u_i\lambda^i\right)\Bigg|_{\lambda=0} = -iqu_1^2\overline{u_0} - 2iqu_0u_2\overline{u_0} - 2iqu_0u_1\overline{u_1} - iqu_0^2\overline{u_2} + ku_2u_{0_x}\overline{u_0} + ku_1u_{1_x}\overline{u_0} + ku_1u_{0_x}\overline{u_1} + ku_0u_{2_x}\overline{u_0} + ku_0u_{1_x}\overline{u_1} + ku_0u_{0_x}\overline{u_2},$$

For u_0 we have,

$$\frac{\partial^\alpha u_0(x,t)}{\partial t^\alpha} = 0;\ \ u_0(x,0) = e^{icx},$$

By taking the left-sided Riemann Liouville integral given by equation (2) for both sides of the last equation and applying the property (6) we obtain,

$$u_0(x,t) = u_0(x,0) = e^{icx}.$$

For u_1 we have,

$$\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = B_1(L_0 + N_0) \quad ;\ u_1(x,0) = 0,$$

$$L_0 = L(u_0) = -ip\frac{\partial^2 u_0(x,t)}{\partial x^2} = ic^2pe^{icx},$$

$$N_0 = -iqu_0^2\overline{u_0} + ku_0u_{0_x}\overline{u_0} = (-q + ck)ie^{icx},$$

$$L_0 + N_0 = i(c^2p - q + ck)e^{icx},$$

$$\frac{\partial^\alpha u_1(x,t)}{\partial t^\alpha} = B_1(i(c^2p - q + ck)e^{icx}) \quad ; u_1(x,0) = 0,$$

By taking the left sided Riemann Liouville integral given by equation (2) for both sides of the last equation and applying properties (4), (5) and (6) we obtain,

$$\frac{1}{\Gamma(\alpha)}\int_0^t \frac{\partial^\alpha u_1(x,s)}{\partial s^\alpha}(t-s)^{\alpha-1}ds = \frac{1}{\Gamma(\alpha)}\int_0^t B_1(i(c^2p - q + ck)e^{icx}) \quad (t-s)^{\alpha-1}ds,$$

Therefore,

$$u_1(x,t) - u_1(x,0) = \left(\frac{iB_1\,t^\alpha(c^2p - q + ck)e^{icx}}{\Gamma(\alpha+1)}\right),$$

$$u_1(x,t) = \frac{iB_1\,t^\alpha(c^2p - q + ck)e^{icx}}{\Gamma(\alpha+1)}$$

Similarly, we obtain

$$u_2(x,t) = \frac{iB_2(c^2p - q + ck)\,t^\alpha\,u_0}{\Gamma(\alpha+1)} - \frac{B_1^2(c^2p - q + ck)^2t^{2\alpha}u_0}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = \frac{i(c^2p - q + ck)B_3\,t^\alpha\,e^{icx}}{\Gamma(\alpha+1)} - \frac{2(c^2p - q + ck)^2B_1B_2t^{2\alpha}e^{icx}}{\Gamma(2\alpha+1)} - \frac{ipc^2(c^2p - q + ck)^2B_1^3t^{3\alpha}e^{icx}}{\Gamma(3\alpha+1)} + \frac{i(-q + ck)(c^2p - q + ck)^2t^{3\alpha}B_1^3e^{icx}\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2\Gamma(3\alpha+1)} - \frac{3i(-q + ck)(c^2p - q + ck)^2B_1^3t^{3\alpha}e^{icx}}{\Gamma(3\alpha+1)},$$

For the special case $\alpha = 1$, we get the general formula for u_n,

$$\frac{\partial^\alpha u_n(x,t)}{\partial t^\alpha} = i(c^2p - q + ck)\sum_{k=0}^{n-1} B_{n-k}u_k(x,t),$$

where $u_n(x,0) = 0$. Therefore, one arrives at

$$u_n(x,t) = i(c^2p - q + ck)\int_0^t \sum_{k=0}^{n-1} B_{n-k}u_k(x,s)\,ds.$$

From this relation we get,

$$u_4(x,t)=\frac{B_4(c^2p-q+ck)\,\mathrm{t}e^{icx}}{\Gamma(1+1)}-\frac{B_2^2(c^2p-q+ck)^2t^2e^{icx}}{\Gamma(1+2)}-\frac{2B_1B_3(c^2p-q+ck)^2t^2e^{icx}}{\Gamma(1+2)}-\frac{3i(c^2p-q+ck)^3B_1^2B_2t^3e^{icx}}{\Gamma(1+3)}+\frac{B_1^4(c^2p-q+ck)^4t^4e^{icx}}{\Gamma(1+4)}.$$

$$u_5(x,t)=\frac{iB_5(c^2p-q+ck)\,\mathrm{t}\,e^{icx}}{\Gamma(1+1)}-\frac{2B_2B_3(c^2p-q+ck)^2t^2e^{icx}}{\Gamma(1+2)}-\frac{2B_1B_4(c^2p-q+ck)^2t^2e^{icx}}{\Gamma(1+2)}-\frac{3iB_1B_2^2(c^2p-q+ck)^3t^3e^{icx}}{\Gamma(1+3)}-\frac{3iB_1^2B_3(c^2p-q+ck)^3t^3e^{icx}}{\Gamma(1+3)}+\frac{4B_1^3B_2(c^2p-q+ck)^4t^4e^{icx}}{\Gamma(1+4)}+\frac{iB_1^5(c^2p-q+ck)^5t^5e^{icx}}{\Gamma(1+5)}.$$

And,

$$u_6(x,t)=\frac{iB_6(c^2p-q+ck)t\,e^{icx}}{\Gamma(1+1)}-\frac{B_3^2t^2(c^2p-q+ck)^2e^{icx}}{\Gamma(1+2)}-\frac{2B_2B_4(c^2p-q+ck)^2t^2e^{icx}}{\Gamma(1+2)}-\frac{2B_1B_5(c^2p-q+ck)^2t^2e^{icx}}{(1+2)}-\frac{iB_2^3(c^2p-q+ck)^3t^3e^{icx}}{\Gamma(1+3)}-\frac{6iB_1B_2B_3(c^2p-q+ck)^3t^3e^{icx}}{\Gamma(1+3)}-\frac{3iB_1^2B_4(c^2p-q+ck)^3t^3e^{icx}}{\Gamma(1+3)}+\frac{6B_1^2B_2^2(c^2p-q+ck)^4t^4e^{icx}}{\Gamma(1+4)}+\frac{4B_1^3B_3(c^2p-q+ck)^4t^4e^{icx}}{\Gamma(1+4)}+\frac{5iB_1^4B_2(c^2p-q+ck)^5t^5e^{icx}}{\Gamma(1+5)}-\frac{B_1^6(c^2p-q+ck)^6t^6e^{icx}}{\Gamma(1+6)}.$$

Then approximate solution is obtained as,

$$\tilde{u}_n(x,t;b_i)=\sum_{i=0}^{n}u_i(x,t). \tag{21}$$

The values of $B_i=b_i; i=1,2,3,\dots$ are determined by the least-squares principle. First, we compute the residual,

$$R(x,t;b_i)=\frac{\partial^\alpha\tilde{u}_n(x,t;b_i)}{\partial t^\alpha}-ip\frac{\partial^2\tilde{u}_n(x,t;b_i)}{\partial x^2}-iq|\tilde{u}_n(x,t;b_i)|^2\tilde{u}_n(x,t;b_i)+k|\tilde{u}_n(x,t;b_i)|^2\frac{\partial\tilde{u}_n(x,t;b_i)}{\partial x}.$$

If the residual R is identically zero throughout the domain and the initial/boundary conditions are satisfied, then the candidate function satisfies the governing problem, otherwise, we choose the values of b_i such that the norm squared of the residual over the entire domain D is minimized:

$$\epsilon(b_1,b_2,b_3,\dots)=\int_{(x,t)\in D}|R(x,t;b_i)|^2dxdt.$$

Due to the nonlinear nature of the resulting parameter-identification problem, MATLAB was used to compute the numerical minimizers of the residual functional. Table 1 shows
the numerical values of the control parameters $b_1,b_2,b_3,\dots$.

Table 1: Values of b_i for $n=3, c=q=0.2, k=p=0.5$ and different values of α

α	b_1	b_2	b_3
1	−0.9999	0.0002	−0.0003
0.9	−0.9969	0.0002	0.0002
0.8	−0.9961	0.0002	0.0002
0.7	−0.9951	0.0002	0.0001

Table 2: Values of b_i for $n=6$, $c=q=0.2, k=p=0.5$ and $\alpha=1$

b_1	b_2	b_3	b_4	b_5	b_6
−1	0	0	0	0	0

In this example, the AHPM was applied for the TFCLLE (15), then the approximate solution (21) was compared with the exact solution for the case $\alpha=1, c=q=0.2, k=p=0.5$, as given by the equation (16).
Table 1 presents the values of the parameters b_j; $j=1,2,\dots,n$ for number of iterations $n=3$ and various values of the fractional order α, whereas Table 2 provides the corresponding parameter values for the case $n=6$ when $\alpha=1$. Moreover, Tables 3 and 4 list selected values of both the exact and the approximate solution with the corresponding absolute error, evaluated at different points within the domain $D=[-2,2]\times[0,1]$ for $n=3$ and $n=6$, respectively.
Figure 1 illustrates the norm of the exact solution (16) and AHPM solution (21) for $\alpha=1, n=3$ over the domain $D=[-2,2]\times[0,1]$. The approximate solution of equation (15) shows a satisfactory agreement with the exact solution. Moreover, Figure 2 depict the behavior of the absolute error for $n=3$ and $n=6$. The absolute error is found to be of the order of 10^{-7} for number of iterations $n=3$ and 10^{-12} for $n=6$. Further, Figure 3 shows the approximate solution for different values of α, where it is noted that as $\alpha\to 1$, the approximate solution converges in a good manner to the exact solution at $\alpha=1$. These results confirm that AHPM is a highly efficient technique for solving the equation under study.

Conclusions

In this study, an asymptotic homotopy perturbation method was developed and applied to the time-fractional Chen–Lee–Liu equation with the Caputo fractional derivative. The nonlinear fractional problem was transformed into a sequence of recursive problems, and the convergence-control parameters were obtained through residual minimization using the least squares principle. The numerical experiments showed that the truncated AHPM series approaches the available exact solution as the truncation order increases. The accuracy was assessed using absolute errors and residual-based measures for selected fractional orders. These results support the applicability of AHPM to the considered model. Further work should investigate its performance against alternative numerical and semi-analytical methods and establish more general convergence and error estimates.

Table 3: Absolute error for the values $n = 3, c = q = 0.2, k = p = 0.5$ and $\alpha = 1$.

x	t	Exact solution $\mathbf{u(x,t)}$	Approximate solution $\mathbf{\tilde{u}_3\,(x,t;b_i\,)}$	Absolute error $\mathbf{\|\|\tilde{u}_3\,(x,t;b_i\,) - u(x,t)\|\|}$
-2	0.001	$0.921092144522 - 0.3893446561i$	$0.921092144526 - 0.3893446561i$	$3.52071 * 10^{-12}$
	0.01	$0.921372233903 - 0.38868136897i$	$0.9213722342 - 0.3886813690i$	$3.62375 * 10^{-10}$
	0.1	$0.9241468337164 - 0.38203747163i$	$0.924146902945 - 0.3820373978i$	$1.01185 * 10^{-7}$
-1	0.001	$0.98008246825 - 0.19859092483i$	$0.980082468254 - 0.198590924i$	$3.52071 * 10^{-12}$
	0.01	$0.9802251996676 - 0.19788521404i$	$0.980225200029 - 0.19788521402i$	$3.62375 * 10^{-10}$
	0.1	$0.98162455357131 - 0.19082252441i$	$0.981624606758 - 0.19082243834i$	$1.01185 * 10^{-7}$
1	0.001	$0.9800506811585 + 0.1987477354i$	$0.9800506811620 + 0.1987477354i$	$3.52071 * 10^{-12}$
	0.01	$0.97990732877227 + 0.1994533203i$	$0.979907329100 + 0.19945332055i$	$3.62375 * 10^{-10}$
	0.1	$0.9784458781847 + 0.2065034224i$	$0.97844589365 + 0.2065035223i$	$1.01185 * 10^{-7}$
2	0.001	$0.9210298375881 + 0.38949202594i$	$0.9210298375913 +0.3894920259i$	$3.52071 * 10^{-12}$
	0.01	$0.920749164622 + 0.39015506641i$	$0.920749164913 + 0.39015506662i$	$3.62375 * 10^{-10}$
	0.1	$0.917916206700 + 0.3967742903i$	$0.91791620199 + 0.3967743914i$	$1.01185 * 10^{-7}$

Table 4: Absolute error for $n = 6,\ c = q = 0.2, k = p = 0.5$ and $\alpha = 1$.

x	t	Exact solution $\mathbf{u(x,t)}$	Approximate solution $\mathbf{\tilde{u}_6(x,t;b_i\,)}$	Absolute error $\mathbf{\|\|\tilde{u}_6(x,t;b_i\,) - u(x,t)\|\|}$
-2	0.001	$0.921092144522 - 0.3893446561i$	$0.9210921445228 - 0.38934465618i$	$8.54293 * 10^{-14}$
	0.01	$0.921372233903 - 0.38868136897i$	$0.9213722339376 - 0.38868136889i$	$8629333 * 10^{-11}$
	0.1	$0.9241468337164 - 0.38203747163i$	$0.924146870685 - 0.38203738419i$	$9.4933333 * 10^{-8}$
-1	0.001	$0.98008246825 - 0.19859092483i$	$0.98008246825149 - 0.19859092483i$	$8.54293 * 10^{-14}$
	0.01	$0.9802251996676 - 0.19788521404i$	$0.980225199684 - 0.19788521395i$	$8629333 * 10^{-11}$
	0.1	$0.98162455357131 - 0.19082252441i$	$0.98162457243165 - 0.19082243137i$	$9.4933333 * 10^{-8}$
1	0.001	$0.9800506811585 + 0.1987477354i$	$0.9800506811585 + 0.1987477354i$	$8.54293 * 10^{-14}$
	0.01	$0.97990732877227 + 0.1994533203i$	$0.979907328755 + 0.19945332048i$	$8629333 * 10^{-11}$
	0.1	$0.9784458781847 + 0.2065034224i$	$0.9784458593243 + 0.206503515422i$	$9.4933333 * 10^{-8}$
2	0.001	$0.9210298375881 + 0.38949202594i$	$0.921029837588 + 0.3894920259i$	$8.54293 * 10^{-14}$
	0.01	$0.920749164622 + 0.39015506641i$	$0.9207491645891 + 0.39015506649i$	$8629333 * 10^{-11}$
	0.1	$0.917916206700 + 0.3967742903i$	$0.9179161697312 + 0.39677437778i$	$9.4933333 * 10^{-8}$

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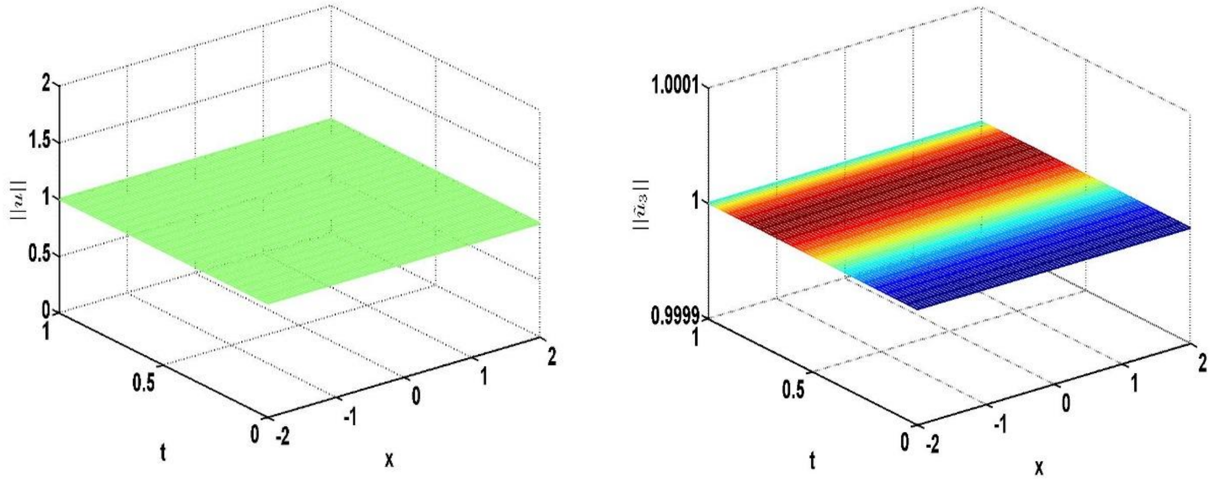

Figure 1: The norm of the exact solution $||u(x,t)||$ (left) and the norm of the AHPM solution $||\tilde{u}_n(x,t;b_i)||$ (right) at $\alpha = 1$, $n = 3$

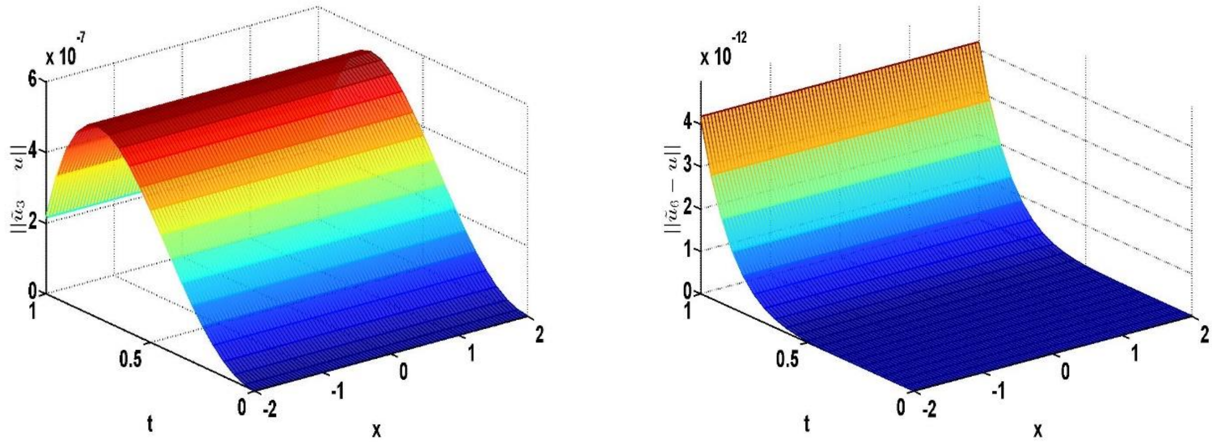

Figure 2: The absolute error $||\tilde{u}_n(x,t;b_i) - u(x,t)||$ at $\alpha = 1$, $n = 3$ (left) and $n = 6$ (right)

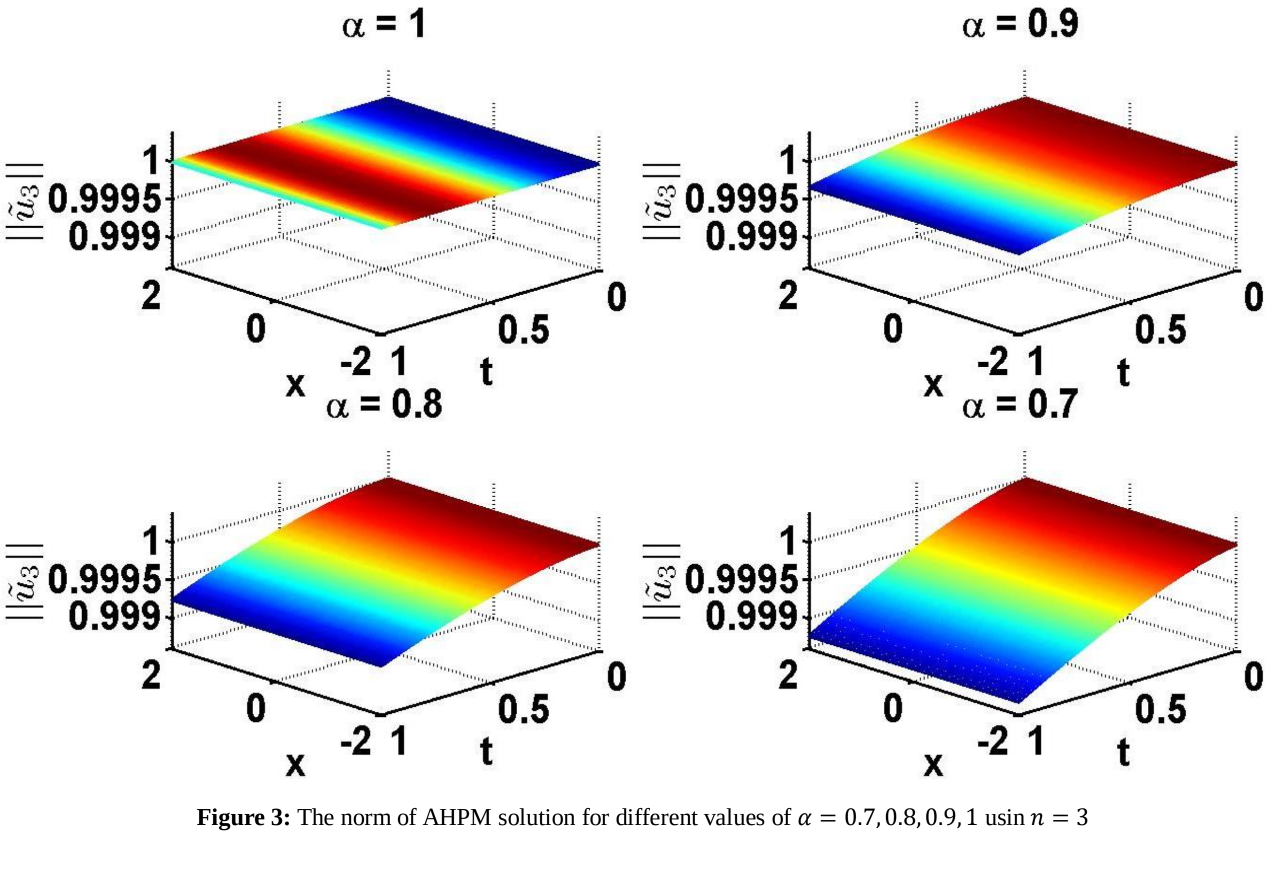

Figure 3: The norm of AHPM solution for different values of $\alpha = 0.7, 0.8, 0.9, 1$ usin $n = 3$

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