Wadi Alshatti University Journal of Pure and Applied Sciences

مجلة جامعة وادي الشاطئ للعلوم البحتة و التطبيقية

Volume 4, No. 2, July-December 2026 Online ISSN: 3006-0877 المجلد 4، الاصدار 2، يوليو - ديسمبر 2026

RESEARCH ARTICLE APPLIED MATHEMATICS

Generalized Birecurrent Finsler Geometry: Curvature Identities and Mathematical Foundations for Mechanical and Renewable Energy Systems

Wafa'a H. A. Hadi[1], Adel Mohammed Ali Al-Qashbari[2,*], Rahma S. Elzer[3]

[1]Departmenr of Mathematics, Community College-Aden, Aden, Yemen
[2]Department of Biomedical Engineering, Faculty of Engineering and Computers, University of Science and Technology, Aden, Yemen
[3]Physics Department, Faculty of Science, Derna University, Derna, Libya.

ARTICLE HISTORY

Received 29 July 2026
Revised 02 September 2026
Accepted 19 September 2026
Online 24 September 2026

KEYWORDS

Finsler Geometry;
Generalized Birecurrent Spaces;
Cartan Connection;
Curvature Identities;
Mechanical Systems;
Renewable Energy Systems.

ABSTRACT

This paper investigates a class of generalized birecurrent Finsler spaces characterized by a generalized recurrence condition imposed on Cartan's third curvature tensor with respect to Cartan's connection. The geometric framework is formulated using non-null covariant vector fields and higher-order covariant tensor fields. Fundamental consequences of the generalized birecurrence condition are derived, including identities involving the h(v)-torsion tensor, deviation tensor, Ricci tensor, curvature vector, and curvature scalar. It is shown that the Ricci tensor, curvature vector, deviation tensor, and curvature scalar are non-vanishing in the considered class of Finsler spaces. Several curvature identities are established through covariant differentiation, tensor contraction, and transvection of the fundamental geometric relations. The developed framework provides a mathematical basis for studying direction-dependent and curvature-constrained dynamical structures. It also suggests potential extensions to mechanical systems and renewable energy technologies involving directional states, nonlinear motion, and geometric constraints. These results establish a theoretical foundation for developing Finsler-geometric models of mechanical dynamics, energy conversion, and renewable energy systems.

الهندسة الفِنْسلرية ثنائية التكرار المعمَّمة: متطابقات الانحناء والأسس الرياضية للأنظمة الميكانيكية و أنظمة الطاقة المتجددة

وفاء هادي أحمد هادي[1]، عادل محمد علي القشبري[2,*]، رحمة سالم الزير[3]

الملخص

في هذا العمل، تم تطوير إطار موحّد عالي المستوى يربط بين الجريان اللاتماثلي وبنى التكرارية من الرتب العليا وهندسة فينسر. ويركز الإطار على العلاقات بين موتر الانحناء الإسقاطي لوايل (Weyl projective curvature tensor) وموتر الانحناء الدائري (concircular curvature tensor) في فضاءات فينسر التكرارية الممتدة من الرتبة الثانية والفضاءات الفينسلرية ثنائية التكرار (birecurrent Finsler spaces). ومن خلال اشتقاق علاقات موترية جديدة وشروط ضرورية وكافية صارمة، جرى توسيع نظرية التكرار الكلاسيكية. وللتحقق من قابلية التطبيق، أُجريت محاكاة باستخدام ديناميكا الموائع الحسابية (CFD) لجريان لزج داخل أنابيب منحنية ذات انحناءات متفاوتة وظروف دخول مختلفة. وأظهرت النتائج اعتمادًا اتجاهيًا ملحوظًا لكلٍّ من السرعة وإجهاد القص على الجدار. كما تتوافق النتائج مع النموذج الهندسي المقترح، مما يشير إلى وجود علاقة وثيقة بين الهندسة التفاضلية المتقدمة والسلوك اللاتماثلي للموائع، مع ما قد يترتب على ذلك من تطبيقات في نماذج الجاذبية المعدّلة وعلم الكونيات.

الكلمات المفتاحية

هندسة فينسر الجريان اللاتماثلي
تفاعل انحناء وايل الدائري
الفضاءات الفينسلرية ثنائية التكرار
التكرار من الرتب العليا
ديناميكا الموائع الحسابية (FD .

Introduction

Finsler geometry constitutes an important extension of Riemannian geometry and provides a flexible mathematical framework for describing geometric structures whose properties depend on both position and direction. This directional dependence makes Finsler geometry particularly suitable for studying nonlinear and anisotropic phenomena. In recent decades, considerable attention has been devoted to recurrent and generalized recurrent structures in Finsler geometry because they provide constrained relationships between geometric quantities and their covariant derivatives. Various forms of recurrence have been investigated in association with different Finsler connections and curvature tensors, leading to a rich class of geometric structures and identities.

Among the fundamental geometric objects in Finsler geometry, Cartan's connection and its associated curvature tensors play a central role in characterizing the intrinsic structure of a Finsler space. Recurrence conditions imposed on curvature tensors provide an effective framework for analyzing the behavior of curvature under covariant differentiation. Previous investigations have considered recurrent, generalized recurrent, projective recurrent, birecurrent, and related higher-order Finsler structures, revealing significant relationships among curvature, torsion, deviation, and Ricci-type tensors. These results provide

*Corresponding author
https://doi.org/10.63318/waujpasv4i2_49

important structural constraints for the characterization of generalized Finsler spaces.

Motivated by these developments, the present work investigates a class of generalized birecurrent Finsler spaces in which Cartan's third curvature tensor satisfies a generalized recurrence condition with respect to Cartan's connection. The proposed formulation is based on non-null covariant vector fields and associated covariant tensor fields. By applying covariant differentiation, contraction, transvection, and Bianchi-type identities, several fundamental relationships are derived for the geometric quantities associated with the considered space. In particular, the study establishes identities involving the $h(v)$-torsion tensor, deviation tensor, Ricci tensor, curvature vector, and curvature scalar.

A principal objective of this study is to characterize the consequences of the generalized birecurrence condition for these fundamental curvature quantities. The analysis shows that the Ricci tensor, curvature vector, deviation tensor, and curvature scalar are non-vanishing within the generalized birecurrent class under consideration. Furthermore, the resulting curvature identities provide additional structural restrictions and extend the collection of known relationships associated with recurrent Finsler geometries.

The significance of the proposed framework also extends to potential applications in mathematical modeling. Because Finsler geometry incorporates directional dependence, it offers a suitable geometric perspective for systems involving nonlinear motion, rotational states, anisotropic behavior, and geometric constraints. In mechanical engineering, such features occur in various dynamical systems and rotating machinery. Similarly, renewable energy technologies [1-4], including wind energy conversion [5,6] and solar-tracking systems [7-9], involve directional motion, orientation, and constrained mechanical processes. Although the present work is theoretical and does not introduce a numerical or experimental model of a specific engineering system, its curvature structure and tensor identities may provide a mathematical foundation for future direction-dependent models of mechanical dynamics and energy-conversion processes.

Accordingly, the main contribution of this paper is the development of a rigorous generalized birecurrent Finsler framework and the derivation of associated curvature identities through Cartan's connection. The obtained results enrich the theoretical structure of generalized recurrent Finsler geometry and provide a basis for future application-oriented studies in nonlinear mechanics, anisotropic systems, and renewable-energy technologies.

Finsler geometry provides a powerful mathematical framework for describing anisotropic geometric structures and has attracted increasing attention because of its applications in mechanics, physics, robotics, and nonlinear dynamical systems [10-12]. The foundational development of Finsler spaces and their differential-geometric structures has been established through classical treatments of Finsler geometry and subsequent studies of curvature, geodesics, and geometric invariants [12,13]. More recent investigations have extended these foundations toward nonlinear spectra, curvature properties, and geometric models with applications in physical systems [10, 14-16]. In the context of recurrent Finsler geometry, several studies have examined recurrent tensor fields and the structural properties of generalized recurrent spaces, including torsion-related tensors and their special geometric conditions [17,18]. The concept of birecurrence has subsequently been investigated for different tensor fields and Finsler structures, providing broader recurrence conditions beyond the classical framework [19]. Higher-order recurrence has also been developed through fourth-order and generalized recurrent Finsler spaces, with particular attention to recurrence decompositions and higher-order curvature tensors [20-22]. Further contributions have addressed BK-5th and BW-four-recurrent Finsler spaces, thereby extending recurrence theory to higher-order geometric structures [23,24]. Studies involving generalized recurrence and Lie derivatives have established additional relationships between curvature tensors and their differential properties [25,26]. Particular attention has also been devoted to projective and M-projective curvature tensors in recurrent and birecurrent Finsler spaces, leading to new geometric identities and characterizations [27,28]. The interaction between generalized recurrence and fundamental curvature tensors, including Cartan and Weyl tensors, has further enriched the geometric characterization of birecurrent Finsler spaces [29]. Recent research has increasingly connected these mathematical structures with applied problems, including anisotropic image processing, where generalized recurrent Finsler models provide a geometric framework for representing direction-dependent structures [30]. Higher-order recurrent Finsler geometry has also been associated with anisotropic flow modelling and computational fluid dynamics, demonstrating its potential in modelling complex mechanical systems [31]. Moreover, generalized recurrent Finsler structures have been investigated in intelligent robotic path planning and anisotropic biomedical systems, highlighting their relevance to emerging engineering applications [32,33]. Related developments in spatial elasticity, nonlinear stress estimation, and natural convection further demonstrate the broader relevance of advanced mathematical modelling to engineering phenomena [34-38]. Despite these advances, the systematic integration of higher-order recurrence conditions, curvature identities, and application-oriented Finsler structures remains an active research direction, motivating further development of generalized recurrent and birecurrent Finsler models.

Despite these significant developments, the existing literature remains largely focused on particular classes of recurrent or birecurrent Finsler spaces and on individual curvature tensors or differential operators. A systematic treatment that combines generalized birecurrence with the derivation of interconnected identities for the principal geometric tensors associated with Cartan's connection is still limited. In particular, the relationships among the $h(v)$-torsion tensor, deviation tensor, Ricci tensor, curvature vector, and curvature scalar under a generalized birecurrent condition require further investigation. The present work addresses this gap by establishing a unified framework for generalized birecurrent Finsler spaces and deriving new curvature identities through covariant differentiation, contraction, transvection, and Bianchi-type relations. Moreover, the non-vanishing character of the fundamental curvature quantities is examined within the proposed structure. Beyond its intrinsic geometric significance, the framework provides a mathematical basis for describing direction-dependent and curvature-constrained phenomena, with potential relevance to anisotropic mechanical systems, nonlinear dynamics, and renewable-energy technologies.

Let F be a Finsler metric defined on an n-dimensional Finsler manifold, satisfying the standard regularity and positivity conditions. Denote by g_{ij} the corresponding fundamental metric tensor, and by G^i_{jk} and G^{*i}_{jk} the coefficients of the Cartan and Berwald connections, respectively.
These connection coefficients are symmetric with respect to their lower indices and are positively homogeneous of degree zero with respect to the directional variables.
The Cartan and Berwald connection coefficients, together with their associated tensorial quantities, are related through the following fundamental relation:

(1.1) $g_{ij}\, g^{jk} = \delta^k_i = \begin{cases} 1 \ , & if \quad i = k \ , \\ 0 \ , & if \quad i \neq k \ . \end{cases}$

The vectors y_i and y^i satisfies the following relations

(1.2) a) $y_i = g_{ij}\, y^j$, b) $y_i\, y^i = F^2$,

c) $g_{ij} = \dot{\partial}_i\, y_j = \dot{\partial}_j\, y_i$,

d) $g_{ij}\, y^j = \frac{1}{2}\dot{\partial}_i F^2 = F\dot{\partial}_i F$,

and e) $\dot{\partial}_j\, y^i = \delta^i_j$.

The tensor C_{ijk} defined by

(1.3) $C_{ijk} = \frac{1}{2}\dot{\partial}_i\, g_{jk} = \frac{1}{4}\dot{\partial}_i\, \dot{\partial}_j\, \dot{\partial}_k\, F^2$,

is known as (h) hv-torsion tensor.
It is positively homogeneous of degree -1 in the directional arguments and symmetric in all its indices.
The (v) hv-torsion tensor C^h_{ik} and its associate (h) hv-torsion tensor C_{ijk} are related by

(1.4) a) $C_{ijk}y^i = C_{kij}y^i = C_{jki}y^i = 0$,

b) $C^i_{jk}\, y^j = C^i_{kj}\, y^j = 0$,

and c) $C^h_{ik} = g^{hj}C_{ijk}$.

The (v) hv-torsion tensor C^h_{ik} is also positively homogeneous of degree -1 in the directional arguments and symmetric in its lower indices.
Cartan deduced the h-covariant derivative for an arbitrary vector filed X^i with respect to x^k

(1.5) $X^i_{|k} = \partial_k\, X^i - \left(\dot{\partial}_r X^i\right) G^r_k + X^r \Gamma^{*i}_{rk}$.

The metric tensor g_{ij} and the vector y^i are covariant constant with respect to a above process, i.e.

(1.6) a) $g_{ij|k} = 0$, b) $y^i_{|k} = 0$ and c) $g^{ij}_{|k} = 0$.

The process of h-covariant differentiation with respect to x^k commute with partial differentiation with respect to y^j for arbitrary vector filed X^i , according to

(1.7) $\dot{\partial}_j\left(X^i_{|k}\right) - \left(\dot{\partial}_j X^i\right)_{|k} = X^r\left(\dot{\partial}_j \Gamma^{*i}_{rk}\right) - \left(\dot{\partial}_r X^i\right) P^r_{jk}$,

where

(1.8) a) $\dot{\partial}_j \Gamma^{*r}_{hk} = \Gamma^{*r}_{jhk}$, b) $P^i_{kh}\, y^k = 0 = P^i_{kh}\, y^h$,

and c) $P^i_{jkh}\, y^j = P^i_{jkh}$.

The tensor P^i_{kh} is called v(hv)-torsion tensor and its associate tensor P_{jkh} is given by

(1.9) a) $g_{rj}\, P^r_{kh} = P_{kjh}$.

The associate tensor P_{ijkh} of the (hv)-curvature tensor P^i_{jkh} is given by

(1.9) b) $g_{ir}\, P^r_{jkh} = P_{ijkh}$.

The quantities H^i_{jkh} and H^i_{kh} form the components of tensors and they called h-curvature tensor of Berwald (Berwald curvature tensor) and h(v)-torsion tensor, respectively, and defined as follow:

(1.10) a) $H^i_{jkh} = \partial_j G^i_{kh} + G^r_{kh}\, G^i_{rj} + G^i_{rhj}\, G^r_k$
$\qquad - \partial_j\, G^i_{hk} - G^r_{hk}\, G^i_{rj} - G^i_{rkj}\, G^r_h$,

and b) $H^i_{kh} = \partial_h G^i_k + G^r_k\, C^i_{rh} - \partial_k G^i_h - G^r_h\, C^i_{rk}$.

They are skew-symmetric in their lower indices, i.e. k and h Also they are positively homogeneous of degree zero and one, respectively in their directional arguments. They are also related by

(1.11) a) $H^i_{jkh}\, y^j = H^i_{kh}$, b) $H^i_{jkh} = \partial_j H^i_{kh}$,

and c) $H^i_{jk} = \partial_j\, H^i_k$.

These tensors were constructed initially by mean of the tensor H^i_h , called the deviation tensor, given by

(1.12) $H^i_h = 2\partial_h G^i - \partial_r G^i_h\, y^r + 2\, G^i_{hs}\, G^s - G^i_s\, G^s_h$.

The deviation tensor H^i_h is positively homogeneous of degree two in the directional arguments.
In view of Euler's theorem on homogeneous functions and by contracting the indices i and h in (1.11) and (1.12), we have the following:

(1.13) a) $H^i_{jk}\, y^j = -\, H^i_{kj}\, y^j = H^i_k$,

and b) $y_i\, H^i_j = 0$.

The quantities H^i_{jkh} and H^i_{kh} are satisfies the following

(1.14) a) $H_{ijkh} = g_{jr}\, H^r_{ihk}$, b) $H_{jk.h} = g_{jr} H^r_{hk}$,

and c) $y_i\, H^i_{jk} = 0$.

Cartan's third curvature tensor R^i_{jkh} satisfies the identity known as Bianchi identity

(1.15) a) $R^i_{jkh|s} + R^i_{jsk|h} + R^i_{jhs|k}$
$\qquad + \left(R^r_{mhs}P^i_{jkr} + R^r_{mkh}P^i_{jsr} + R^r_{msk}P^i_{jhr}\right) y^m = 0$,

and b) $R_{ijhk} + R_{ihkj} + R_{ikjh}$
$\qquad + \left(C_{ijs}\, K^s_{rhk} + C_{ihs}\, K^s_{rkj} + C_{iks}\, K^s_{rjh}\right) y^r = 0$,

where c) $P^r_{ijs} = \dot{\partial}_s \Gamma^{*r}_{ij} - C^r_{is|j} + C^r_{im}\, C^m_{js|k}\, y^k$.

The Ricci tensor R_{jk} , the deviation tensor R^r_h and the curvature scalar R of the curvature tensor R^i_{jkh} are given by

(1.16) a) $R^i_{jkh}\, y^j = H^i_{hk} = K^i_{jhk}\, y^j$,

b) $R_{ijhk} = g_{rj}\, R^r_{ihk}$,

c) $R_{jkhm}\, y^j = H_{kh.m}$, d) $R^r_{ihk} = g^{jr} R_{ijhk}$,

and e) $R^i_{jkh}\, g^{jk} = R^i_h$.

The contracted tensor R_{kh}(Ricci tensor) and R_k (Curvature vector) are also connected by

(1.17) a) $R_{jk}y^k = R_j$, b) $R_{jk}\, y^j = H_k$,

and c) $R^i_{jki} = R_{jk}$.

Also, this tensor satisfies the following relation too

(1.18) a) $R^i_{jkh} = K^i_{jkh} + C^i_{js}\, K^s_{rhk}\, y^r$,

b) $R_{ijkh} = K_{ijkh} + C_{ijs}\, H^s_{kh}$

and c) $R_{jkhm}\, y^j = H_{kh.m}$.

where $R_{ijkh}$is the associate curvature tensor of R^i_{jkh} .
Cartan's fourth curvature tensor K^i_{jkh} and its associate curvature tensor of K_{ijkh} satisfy the following known as Bianchi identities

(1.19) a) $K^i_{jkh} + K^i_{hjk} + K^i_{khj} = 0$,

and b) $K_{jrkh} + K_{hrjk} + K_{krhj} = 0$.

2. On Generalized R^h-Recurrent Space

Consider an n-dimensional Finsler space $F_n$whose Cartan's third curvature tensor satisfies the following generalized birecurrence condition:

(2.1) $R^i_{jkh|m} = \lambda_m R^i_{jkh} + \mu_m\left(\delta^i_h\, g_{jk} - \delta^i_k\, g_{jh}\right)$
$\qquad + \frac{1}{4}\gamma_m\left(R^i_k g_{jh} - R^i_h g_{jk}\right)$, $R^i_{jkh} \neq 0$,

where λ_m and μ_m are non-null covariant vectors field. We shall call such space as a generalized R^h-recurrent space. We shall denote it briefly by GR^h-RF_n .
By taking the covariant derivative of equation (2.1) with

respect to x^l in the framework of Cartan's connection and applying condition (1.6a), we obtain the following relation:

(2.13) $$R^i_{jkh|m|l} = a_{ml}R^i_{jkh} + b_{ml}(\delta^i_k g_{jh} - \delta^i_h g_{jk}) + \frac{1}{4}c_{ml}(R^i_k g_{jh} - R^i_h g_{jk}) + \frac{1}{4}\gamma_m(R^i_k g_{jh} - R^i_h g_{jk})_{|l} .$$

Such that $a_{ml} = \lambda_{m|l} + \lambda_m\lambda_l$, $b_{ml} = \mu_{m|l} + \lambda_m\mu_l$ and $c_{ml} = \lambda_m\gamma_l + \gamma_{m|l}$, where a_{ml}, b_{ml} and c_{ml} are non-vanishing second-order covariant tensor fields, and λ_m is a non-vanishing first-order covariant vector field.

Definition 2.1. A Finsler space in which Cartan's third curvature tensor R^i_{jkh} satisfies condition (2.13) is called a generalized $R_{|h}$-birecurrent space, and the associated tensor is referred to as a generalized h-birecurrent tensor. These are denoted concisely by $G^{2nd}R_{|h}$ - BRF_n and $G^{2nd}h$ - BR, respectively.

Thus, for an n-dimensional Finsler space F_n, if Cartan's third curvature tensor R^i_{jkh} satisfies both conditions (2.12) and (2.13), the corresponding spaces are denoted by $G^{2nd}R_{|h}$-RF_n and $G^{2nd}R_{|h}$- BRF_n, respectively.

Transvecting of (2.1) by the metric tensor g_{ip}, using (1.6a), (1.16b) and in view of (1.1), we get

(2.2) $$R_{jpkh|l} = \lambda_l R_{jpkh} + \mu_l(g_{hp}\, g_{jk} - g_{kp}\, g_{jh}) .$$

Conversely, the transvection of the condition (2.2) by the associate tensor g^{ip} of the metric tensor g_{ip}, yields the condition (2.1). Thus, the condition (2.2) is equivalent to the condition (2.1). Therefore, a generalized R^h-recurrent space characterized by the condition (2.2).

Therefore, we have

Theorem 2.1. An GR^h - RF_n may characterized by the condition (2.2).

Let us consider GR^h- RF_n characterized by condition (2.2).

Transvecting the condition (2.1) by y^j, using (1.6b), (1.16a) and (1.2a), we get

(2.3) $$H^i_{kh|l} = \lambda_l H^i_{kh} + \mu_l(\delta^i_h\, y_k - \delta^i_k y_h) .$$

Further, transvecting (2.3) byy^k, using (1.6b), (1.13a), (1.2b) and in view of (1.1), we get

(2.4) $$H^i_{h|l} = \lambda_l H^i_h + \mu_l(\delta^i_h F^2 - y_h\, y^i) .$$

Therefore, we have

Theorem 2.2. In GR^h-RF_n, the h-covariant derivative of the h(v)-torsion tensor H^i_{kh} and the deviation tensor H^i_h is given by the conditions (2.3) and (2.4), respectively.

Contracting the indices i and h in the condition (2.1), using (1.17c) and (1.1), we get

(2.5) $$R_{jk|l} = \lambda_l R_{jk} + (n-1)\,\mu_l g_{jk} .$$

Transvecting (2.5) by y^k, using (1.6b), (1.17a) and (1.2a), we get

(2.6) $$R_{j|l} = \lambda_l R_j + (n-1)\,\mu_l y_j .$$

Further, transvecting the condition (2.1) by the associate tensor g^{jk} of the metric tensor g_{jk}, using (1.6c), (1.16e) and in view of (1.1), we get

(2.7) $$R^i_{h|l} = \lambda_l\, R^i_h + (n-1)\,\mu_l\delta^i_h .$$

Contracting the indices i and h in condition (2.7) and using (1.1), we get

(2.8) $$R_{|l} = \lambda_l\, R + n\,(n-1)\,\mu_l .$$

where $R^r_r = R$.

The conditions (2.5), (2.6), (2.7) and (2.8), show that the Ricci tensor R_{jk}, the curvature vector R_j, the deviation tensor R^i_h and the curvature scalar R of a generalized R^h-recurrent space cannot vanish, because the vanishing of them imply the vanishing of the covariant vector field μ_l, i.e. $\mu_l = 0$, a contradiction.

Thus, we conclude

Theorem 2.3. In GR^h - RF_n, the Ricci tensor R_{jk}, the curvature vectorR_j, the deviation tensor R^i_h and the curvature scalar R are non-vanishing.

Certain Identities

In this section we shall obtain some identities in GR^h-RF_n.

Taking h- covariant differentiation of the formula (1.15b) with respect to x^l in the sense of Cartan and transvecting (1.15b) by the associate tensor g^{jp} of the metric tensor g_{jp}, using (1.6c), (1.16a), (1.16d) and (1.4a), we get

(3.1) $$R^p_{ihk|l} + g^{jp}\, R_{ihkj|l} + g^{jp}\, R_{ikjh|l} + (C^p_{is}H^s_{hk} + g^{jp}C_{ihs}\, H^s_{kj} + g^{jp}C_{iks}\, H^s_{jh})_{|l} = 0 .$$

Transvecting (3.1) by y^i, using (1.6b), (1.16a), (1.18c), (1.4b)) and (1.4a), we get

(3.2) $$H^p_{hk|l} + g^{jp}\, H_{hk.j|l} + g^{jp}\, H_{kj.h|l} = 0 .$$

Thus, we conclude

Theorem 3.1. In GR^h-RF_n, the identities (3.1) and (3.2) hold good.

Using (1.16b) and (1.16a) in the identity (1.15b), we get

(3.3) $$g_{rj}R^r_{ihk} + g_{rh}R^r_{ikj} + g_{rk}R^r_{ijh} + C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} = 0 .$$

Now, transvecting (3.3) by y^i, using the equations (1.16a) and (1.4b), we get

(3.4) $$g_{rj}\, H^r_{hk} + g_{rh}\, H^r_{kj} + g_{rk}\, H^r_{jh} = 0 .$$

By using (1.14b), the equation (3.4) yields to

(3.5) $$H_{hj.k} + H_{kh.j} + H_{jk.h} = 0 .$$

Transvecting (3.4) by y^h, using (1.13a) and (1.2a), we get

(3.6) $$g_{rj}\, H^r_k = g_{rk}\, H^r_j .$$

Thus, we conclude

Theorem 3.2. In GR^h-RF_n, the identities (3.4), (3.5) and (3.6) hold good.

Using (1.16a) in the identity (1.15a), we get

(3.7) $$R^r_{ijk|h} + R^r_{ihj|k} + R^r_{ikh|j} + H^s_{kh}\, P^r_{ijs} + H^s_{jk}\, P^r_{ihs} + H^s_{hj}\, P^r_{iks} = 0 .$$

In view of the condition (2.1), the identity (3.7), may be written as

(3.8) $$\lambda_h\, R^r_{ijk} + \lambda_k\, R^r_{ihj} + \lambda_j\, R^r_{ikh} + \mu_h(\delta^r_k g_{ij} - \delta^r_j g_{ik}) + \mu_k(\delta^r_j g_{ih} - \delta^r_h g_{ij}) + \mu_j(\delta^r_h g_{ik} - \delta^r_k g_{ih}) + (H^s_{kh}P^r_{ijs} + H^s_{jk}P^r_{ihs} + H^s_{hj}P^r_{iks}) = 0 .$$

Transvecting (3.8) by y^i, using (1.16a), (1.2a) and (1.8c), we get

(3.9) $$\lambda_h\, H^r_{jk} + \lambda_k\, H^r_{hj} + \lambda_j\, H^r_{kh} + \mu_h(\delta^r_k y_j - \delta^r_j y_k) + \mu_k(\delta^r_j y_h - \delta^r_h y_j) + \mu_j(\delta^r_h y_k - \delta^r_k\, y_h) + (H^s_{kh}\, P^r_{js} + H^s_{jk}\, P^r_{hs} + H^s_{hj}\, P^r_{ks}) = 0 .$$

Transvecting (3.9) by y^j, using (1.13a), (1.2b), (1.1) and (1.8b), we get

(3.10) $$\lambda_h\, H^r_k - \lambda_k\, H^r_h + \lambda\, H^r_{kh} + \mu_h(\delta^r_k F^2 - y_k\, y^r) + \mu_k(y_h\, y^r - \delta^r_h F^2) + \mu\,(\delta^r_h y_k - \delta^r_k\, y_h) + (H^s_k\, P^r_{hs} - H^s_h\, P^r_{ks}) = 0 ,$$

where $\lambda_j\, y^j = \lambda$ and $\mu_j\, y^j = \mu$.

Thus, we conclude

Theorem 3.3. In GR^h-RF_n, the identities (3.8), (3.9) and (3.10) hold good.

Further, transvecting (3.9) and (3.10) by the vector y_r , using (1.14c), (1.1), (1.13b) and (1.2b), we get

(3.11) $\left(H^s_{kh}\, P^r_{js} + H^s_{jk}\, P^r_{hs} + H^s_{hj}\, P^r_{ks}\right) y_r = 0$, and

(3.12) $H^s_k\, y_r P^r_{hs} = H^s_h\, y_r\, P^r_{ks}$., respectively.

Thus, we conclude

Theorem 3.4. In GR^h-RF_n , the identity (3.11) holds good.

Theorem 3.5. In GR^h-RF_n , we have the identity (3.12).

Transvecting (3.8), (3.9) and (3.10) by the metric tensor g_{rm} , using (1.16b), (1.1), (1.9b), (1.14b), (1.9a) and (1.2a), we get

(3.13) $$\begin{aligned}&\lambda_h R_{imjk} + \lambda_k R_{imhj} + \lambda_j R_{imkh}\\ &+ \mu_h\left(g_{km}\, g_{ij} - g_{jm}\, g_{ik}\right)\\ &+ \mu_k\left(g_{jm}\, g_{ih} - g_{hm}\, g_{ij}\right)\\ &+ \mu_j\left(g_{hm}\, g_{ik} - g_{km}\, g_{ih}\right)\\ &+ \left(H^s_{kh}\, P_{imjs} + H^s_{jk}\, P_{imhs} + H^s_{hj}\, P_{imks}\right) = 0 \end{aligned}$$,

(3.14) $$\begin{aligned}&\lambda_h\, H_{jm.k} + \lambda_k\, H_{hm.j} + \lambda_j\, H_{km.h}\\ &+\left(H^s_{kh}\, P_{jms} + H^s_{jk}\, P_{hms} + H^s_{hj}\, P_{kms}\right) = 0\end{aligned}$$, and

(3.15) $$\begin{aligned}&g_{rm}\left(\lambda_h H^r_k - \lambda_k\, H^r_h + \lambda\, H^r_{kh}\right)\\ &+ \mu_h\left(g_{km}\, F^2 - y_k\, y_m\right)\\ &+ \mu_k\left(y_h\, y_m - g_{hm}\, F^2\right) + \mu\left(g_{hm}\, y_k - g_{km}\, y_h\right)\\ &+ \left(H^s_k\, P_{hms} - H^s_h\, P_{kms}\right) = 0\end{aligned}$$, respectively.

Thus, we conclude

Theorem 3.6. In GR^h-RF_n , the identities (3.13), (3.14) and (3.15) hold good.

Using (1.18b) and (1.16a) in the identity (1.15b), we have

(3.16) $$\begin{aligned}&K_{ijhk} + K_{ihkj} + K_{ikjh}\\ &+2\left(C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh}\right) = 0\end{aligned}$$.

In view of (1.19b), the identity (3.16) becomes

(3.17) $C_{ijs}\, H^s_{hk} + C_{ihs}\, H^s_{kj} + C_{iks}\, H^s_{jh} = 0$.

Taking h-covariant differentiation of the identity (3.17), with respect to x^l in the sense of Cartan, we get

(3.18) $$\begin{aligned}&C_{ijs|l} H^s_{hk} + C_{ijs}\, H^s_{hk|l} + C_{ihs|l} H^s_{kj} + C_{ihs}\, H^s_{kj|l}\\ &+ C_{iks|l}\, H^s_{jh} + C_{iks}\, H^s_{jh|l} = 0\end{aligned}$$.

Now, transvecting (3.18) by the associate tensor g^{rj}, using (1.6c) and (1.4c), we get

(3.19) $$\begin{aligned}&C^r_{is|l}\, H^s_{hk} + C^r_{is}\, H^s_{hk|l} + C_{ihs|l}\, H^s_{kj}\, g^{rj}\\ &+ C_{ihs}\, H^s_{kj|l}\, g^{rj} + C_{iks|l}\, H^s_{jh}\, g^{rj}\\ &+ C_{iks}\, H^s_{jh|l}\, g^{rj} = 0\end{aligned}$$.

Transvecting (3.18) and (3.19) by y^h, using (1.6b), (1.13a) and (1.4a), we get

(3.20) $C_{ijs|l}\, H^s_k + C_{ijs}\, H^s_{k|l} = C_{iks|l}\, H^s_j + C_{iks}\, H^s_{j|l}$

and

(3.21) $C^r_{is|l}\, H^s_k + C^r_{is}\, H^s_{k|l} = \left(C_{iks|l}\, H^s_j + C_{iks}\, H^s_{j|l}\right) g^{rj}$,

respectively. Thus, we conclude

Theorem 3.7. In GR^h-RF_n , the identities (3.18), (3.19), (3.20) and (3.21) hold good.

Further, transvecting (3.18) and (3.19) by g_{rs} , using (1.6a) and (1.14b), we get

(3.22) $$\begin{aligned}&C_{ijs|l}\, H_{hr.k} + C_{ijs}\, H_{hr.k|l} + C_{ihs|l}\, H_{kr.j}\\ &+ C_{ihs}\, H_{kr.j|l} + C_{iks|l}\, H_{jr.h} + C_{iks} H_{jr.h|l} = 0\end{aligned}$$, and

(3.23) $$\begin{aligned}&C^r_{js|l}\, H_{hr.k} + C^r_{js}\, H_{hr.k|l} + \big(C_{ihs|l}\, H_{kr.j} + C_{ihs}\, H_{kr.j|l}\\ &+ C_{iks|l}\, H_{jr.h} + C_{iks}\, H_{jr.h|l}\big)\, g^{rj} = 0\end{aligned}$$, respectively.

Thus, we conclude

Theorem 3.8. In GR^h-RF_n , the identities (3.22) and (3.23) hold good.

Conclusion

This study has developed a generalized birecurrent Finsler framework based on Cartan's connection and the generalized recurrence condition imposed on Cartan's third curvature tensor. Within this framework, several fundamental consequences of the proposed geometric structure have been established using covariant differentiation, tensor contraction, transvection, and Bianchi-type identities. In particular, new relationships involving the $h(v)$-torsion tensor, deviation tensor, Ricci tensor, curvature vector, and curvature scalar have been derived. The analysis further demonstrates the non-vanishing character of the principal curvature quantities considered in the generalized birecurrent setting. These results provide additional structural constraints for the characterization of generalized recurrent Finsler spaces and contribute to the development of curvature-based identities in Finsler geometry. Overall, the proposed framework extends existing recurrent and birecurrent structures and provides a rigorous mathematical foundation for further investigations of direction-dependent geometric and dynamical systems.

Future Applications and Research Directions

The theoretical framework developed in this study opens several promising directions for future research. First, the generalized birecurrent curvature structure can be incorporated into mathematical models of nonlinear mechanical systems, particularly those involving directional dependence, rotational motion, anisotropic behavior, and geometric constraints. Second, the framework may be investigated in wind-energy conversion systems, where aerodynamic direction, rotor dynamics, and nonlinear mechanical motion interact within a direction-dependent operating environment. Third, Finsler-geometric formulations may be developed for solar-tracking mechanisms, in which the orientation and trajectory of energy-collection surfaces are governed by directional and geometric constraints. Further extensions could also address anisotropic materials, vibration and rotating machinery, nonlinear control systems, and energy-conversion processes.

An important future objective is to establish explicit mathematical models that couple the curvature identities derived here with the governing equations of mechanical and renewable-energy systems. Numerical simulations, stability analysis, and parameter estimation could then be employed to determine whether generalized birecurrent Finsler structures provide measurable advantages over conventional Riemannian or Euclidean formulations. Such developments would transform the present theoretical framework into an application-oriented geometric modeling methodology for direction-dependent mechanical dynamics and renewable-energy technologies.

Author contributions: Hadi and **Al-Qashbari**: Conceptualization; investigation, validation, writing-original draft preparation; **Elzer**: Review and editing, formal analysis. All authors have read and agreed to the published version of the manuscript.

Data availability statement: "The datasets generated during the current study are not publicly available as the study is on-going; however, they are available from the corresponding author upon reasonable request."

Declaration of conflict of interest: "The authors confirm that they have no conflicts of interest to declare."

Funding information: "No funding is received for this work."

References

[1] M. Fakher, et al. "Assessing renewable energy penetration in the Libyan power sector." *Discover Sustainability*, 2026. https://doi.org/10.1007/s43621-026-04483-0

[2] M. Mohamed, et al.."Comprehensive and Concise Guide on Economic Dispatch of Electrical Power Systems Up to Artificial Intelligence Domain." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 2, pp. 398-417, 2026. https://doi.org/10.63318/waujpasv4i2_46

[3] A. Aisa, et al. "Design and Analysis of an off-Grid PV/Wind/Battery/Diesel Generator Hybrid Renewable Power System for a Healthcare Facility in Gharyan, Libya." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 2, pp. 68-91, 2026. https://doi.org/10.63318/waujpasv4i2_09

[4] E. Hala, et al. "Analysis of a grid/wind/hydrogen hybrid energy system for producing electricity, vehicle fuel and freshwater for Gaza Strip-Palestine." *Fuel*, vol. 428, p. 140358, 2027. https://doi.org/10.1016/j.fuel.2026.140358

[5] A. Elmabruk, M. Salem, M., Khaleel, and A. Mansour. "Prediction of Wind Energy Potential in Tajoura and Mislata Cities." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 125-131, 2025. https://doi.org/10.63318/waujpasv3i2_17

[6] M. Elnaggar, et al. "Leveraging Wind Energy for Electricity and Hydrogen Production: A Sustainable Solution to Power Shortages and Eco-Friendly Alternative Fuel." *Advanced Energy and Sustainability Research*, vol. 7, no. 1, p. e202500049, 2026. https://doi.org/10.1002/aesr.202500049

[7] D. Albuzia, A. Ali, M. Mohmed, and A. Hafez. "Reliable and Robust Optimal Interleaved Boost Converter Interfacing PhotoVoltaic Generator." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 3, no. 2, pp. 192-201, 2025. https://doi.org/10.63318/waujpasv3i2_24

[8] W. Salah, et al. "Grid Reliability Enhancement via PV-Diesel Hybrid System in Palestine: Load Flow Analysis and CO_2 Assessment." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 1, pp. 436-452, 2026. https://doi.org/10.63318/waujpasv4i1_48

[9] E. Salim, et al. "Performance Evaluation of a Standalone PV-Hydrogen Power System for Continuous Supply Reliability in Brack Al-Shatti." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 1, pp. 362-376, 2026. https://doi.org/10.63318/waujpasv4i1_40

[10] C. Li, H. Koibuchi, and V. Egorov. "Nonlinear spectrums of Finsler manifolds." *Mathematische Zeitschrift*, vol. 300, pp. 81-123, 2021. https://doi.org/10.1007/s00209-021-02767-x.

[11] S. Ohta. *Comparison Finsler geometry*. Springer, 2021. https://doi.org/10.1007/978-3-030-80650-7.

[12] H. Rund. *The differential geometry of Finsler spaces*. Springer-Verlag, 1959.

[13] M. Xu. "Geodesic orbit spheres and constant curvature in Finsler geometry." *Differential Geometry and its Applications*, vol. 61, pp. 197-206, 2018. https://doi.org/10.1016/j.difgeo.2018.07.002.

[14] C. Li. "Curvatures and hyperbolic flows for natural mechanical systems in Finsler geometry." *Journal of Dynamical and Control Systems*, vol. 27, 2021. https://doi.org/10.1007/s10883-021-09533-6

[15] B. Yashwanth, S. Murthy, and Z. Nekouee. "Generalized Finslerian wormhole models in f(R,T) gravity." *Particles*, vol. 7, no. 3, pp. 747-767, 2023. https://doi.org/10.3390/particles7030043.

[16] J. Zhu. and B.-Q. Ma. "Lorentz violation in Finsler geometry." *Symmetry*, vol. 15, no. 5, Article 978, 2023. https://doi.org/10.3390/sym15050978.

[17] A. Abdallah, A. Navlekar, and K. Ghadle. "On special spaces of torsion tensor in generalized recurrent Finsler space." *Bulletin of Pure & Applied Sciences Mathematics and Statistics*, vol. 41, no. 1, pp. 74-80, 2022. https://doi.org/10.5958/2320-3226.2022.00011.X.

[18] A. Adel. "Recurrence decompositions in Finsler space." *Journal of Mathematical Analysis and Modeling*, vol. 1, no. 1, pp. 77-86, 2020. https://doi.org/10.48185/jmam.v1i1.40.

[19] A. Abdallah, A. Navlekar, and A. Hamoud. "On birecurrent for some tensors in various Finsler spaces." *Journal of Finsler Geometry and Its Applications*, vol.4, pp.33-44, 2023. https://doi.org/10.22098/jfga.2023.12052.1083.

[20] A. Adel. "Finsler space of fourth order on some properties of R^h-generalized recurrent." *Journal of Science and Technology*, vol. 28, no. 2, pp. 14-22, 2023. https://doi.org/10.20428/jst.v28i2.2034.

[21] A. Mohammed, and A. Mubark. "Decomposition of generalized recurrent tensor fields of R^h-nth order in Finsler manifolds." *Journal of Science and Technology*, vol. 30, no. 2, pp. 11-27, 2025. https://doi.org/10.20428/jst.v30i2.2664.

[22] M. Adel, and A. Mubark. "A study of recurrent Finsler spaces of higher order with Cartan's curvature tensor." *University of Aden Journal of Natural and Applied Sciences*, vol. 27, no. 2, pp. 291-300, 2023. https://doi.org/10.47372/uajnas.2023.n2.a10.

[23] A. Al-Qashbari, and S. Baleedi. "Study on Generalized BK-5th Recurrent Finsler Space." *Computational Mathematics and its Applications*, vol. 1, no. 1, pp. 1-14, 2024. https://doi.org/10.17352/cma.000002.

[24] A. Al-Qashbari and M. Abdullah. "On Generalized BW-Four-Recurrent Finsler Spaces and Their Role in the Geometric Modelling of Intelligent Robotic Systems." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol. 4, no. 2, pp. 350-358, 2026. https://doi.org/10.63318/waujpasv4i2-40.

[25] A. Al-Qashbari, and A. Saeed. "On generalized $K_{(B)}$ quad-recurrent in Finsler space." *Electronic Journal of University of Aden for Basic and Applied Sciences*, vol.5, no.2, pp.173–179, 2024. https://doi.org/10.47372/ejua-ba.2024.2.350.

[26] M. Adel, and S. Baleedi. "On the Lie derivative of curvature tensors and their relations in GBK-5RF$_n$." *Electronic Journal of University of Aden for Basic and Applied Sciences*, vol. 5, no. 4, pp. 464-468, 2024. https://doi.org/10.47372/ejua-ba.4.403.

[27] M. Adel. "A study of the M-projective curvature tensor in generalized recurrent and birecurrent Finsler spaces." *Journal of Science and Technology*, vol.30, no.6, pp.16-25, 2025. https://doi.org/10.20428/jst.v30i6.2917

[28] A. Adel, S. Saleh, and I. Ibedou. "On some relations of R-projective curvature tensor in recurrent Finsler space." *Journal of Nonlinear Modelling and Analysis*, vol. 6, no. 4, pp. 1216-1227, 2024. https://doi.org/ 10.12150/jnma.2024.1216.

[29] A. Al-Qashbari, and F. Al-ssallal. "A Study of Cartan's and Weyl's curvature tensors in the context of generalized birecurrent Finsler geometry." *University of Aden Journal of Natural and Applied Sciences*, vol. 29, no. 1, pp. 43-53, 2025. https://doi.org/10.47372/uajnas.2025. n1.a05.

[30] A. Al-Qashbari, and W. Al-Arashi. "Generalized H^h-recurrent Finsler geometry with applications to anisotropic image processing." *Electronic Journal of University of Aden for Basic and Applied Sciences*, vol. 6, no. 3, pp. 190-200, 2025. https://doi.org/10.47372/ejua-ba.2025.3.458.

[31] B. Muthanna, et al. "Higher-Order Recurrent Finsler Geometry and Anisotropic Flow Modelling with CFD Applications." *Wadi Alshatti University Journal of Pure and Applied Sciences*, vol.4, no.2, pp.323-330, 2026. https://doi.org/10.63318/waujpasv4i2-37.

[32] A. Al-Qashbari, M. Abdullah, and I. Attitalla. "Generalized BP-Recurrent Finsler Geometry with Applications to Intelligent Robotic Path Planning and Anisotropic Biomedical Systems." *Biomedicine and Chemical Sciences*, vol. 5, no. 3, pp. 1-8, 2026.

[33] N. Ratliff, K. Van Wyk, M. Xie, A. Li, and M. Rana. "Generalized nonlinear and Finsler geometry for robotics." 2020. https://doi.org/10.1109/ICRA48506.2021.9561543.

[34] A. Hadoush. "Internally Balanced Spatial Elasticity Tensor." *Wadi Alshatti University Journal of Pure and Applied*

Sciences, vol. 3, no. 2, pp. 98-101, 2025. https://doi.org/10.63318/waujpasv3i2-11.

[35] A. Habeb, and A. Alashlam. "Numerical Solution of Natural Convection in Vertical Channels." *Wadi Alshatti University Journal of Pure and Applied Sciences,* vol. 3, no. 2, pp. 138-148, 2025. https://doi.org/10.63318/waujpasv3i2-18.

[36] S. AbuShanab, and A. Hadoush. "Hardware Implementation of Newton-Raphson Method for Nonlinear Stress Estimation." *Wadi Alshatti University Journal of Pure and Applied Sciences,* vol. 3, no. 2, pp. 244-250, 2025. https://doi.org/10.63318/waujpasv3i2-30.

[37] A. Mohammed, and F. Al-Sallal. "A study of curvature tensors by using Berwald's and Cartan's higher-order derivatives in Finsler spaces." *Technological Applied and Humanitarian Academic Journal,* vol. 1, no. 1, pp. 11-27, 2024.

[38] A. Al-Qashbari. "Generalized R^h-recurrent spaces in Finsler geometry: Properties, identities, and characterizations." *Journal of the Faculties of Education - University of Aden,* vol. 18, no. 2, pp. 621-629, 2024. https://doi.org/ 10.47372/jef.18.2.89.